

ARIKI-KOIKE ALGEBRAS WITH SEMISIMPLE BOTTOMS

JIE DU AND HEBING RUI

School of Mathematics, University of New South Wales
Sydney, 2052, Australia

Department of Mathematics, University of Shanghai for Science & Technology
Shanghai, 200093, P.R.China

Consider the cyclic group C_m of order m and the wreath product $W = W_m^r = C_m \wr \mathfrak{S}_r$, where \mathfrak{S}_r is the symmetric group on r letters. Then the direct product $C = C_m \times \cdots \times C_m$ of r copies C_m is normal in W and sits at the “bottom” of W . Let F be a splitting field of C in which m is not zero. Then the “bottom” FC of FW is semisimple and simple C -modules $N_{\mathbf{i}}$ are indexed by the set $I(m, r)$ of all r -tuples $\mathbf{i} = (i_1, \dots, i_r)$ with $1 \leq i_j \leq m$. Moreover, there is a central primitive idempotent decomposition $1 = \sum_{\mathbf{i} \in I(m, r)} e_{\mathbf{i}}$ such that $N_{\mathbf{i}} \cong e_{\mathbf{i}}FC$. The symmetric group \mathfrak{S}_r acts on $I(m, r)$ by place permutation, and the set of W -orbits is identified with the set $\Lambda(m, r)$ of compositions of r with m parts. Thus \mathbf{i} is in the orbit $\lambda \in \Lambda(m, r)$, denoted $\text{wt}(\mathbf{i}) = \lambda$, if $\lambda_j = \#\{i_k : i_k = j\}$ for all j . Notice that, if $\text{wt}(\mathbf{i}) = \text{wt}(\mathbf{j})$, then we have isomorphism of induced modules: $N_{\mathbf{i}} \uparrow^W \cong N_{\mathbf{j}} \uparrow^W$. Therefore, putting $N_{\lambda} = N_{\mathbf{i}}$ and $e_{\lambda} = e_{\mathbf{i}}$ if $\text{wt}(\mathbf{i}) = \lambda$, we have right W -module isomorphism

$$FW \cong \bigoplus_{\lambda \in \Lambda(m, r)} (N_{\lambda} \otimes_{FC} FW)^{\oplus d_{\lambda}} \cong \bigoplus_{\lambda \in \Lambda(m, r)} (e_{\lambda}FW)^{\oplus d_{\lambda}}.$$

Standard results will give a Morita equivalence between the categories of FW -modules and $eFWe$ -modules, where $e = \sum_{\lambda \in \Lambda(m, r)} e_{\lambda}$. Since $e_{\lambda}FWe_{\mu} \cong \delta_{\lambda\mu} F\mathfrak{S}_{\lambda}$, where \mathfrak{S}_{λ} is the Young subgroup corresponding to λ , we have Morita equivalence

$$(1) \quad FW\text{-mod} \stackrel{\text{Morita}}{\sim} (\oplus_{\lambda \in \Lambda(m, r)} F\mathfrak{S}_{\lambda})\text{-mod}.$$

On the other hand, for the Hecke algebra of type B (i.e., the Hecke algebra associated to the group W_2^r), Dipper and James established a Morita equivalence analogous to (1) in [DJ2].

This paper is going to generalize these results to the Ariki-Koike algebra $\mathbf{H} = \mathbf{H}_m^r$, an Iwahori-Hecke type algebra associated with W_m^r (see [AK]). A major difficulty here is the non-existence of a subalgebra based on the bottom C , comparing

The research was carried out while the second author was visiting the University of New South Wales. Both authors gratefully acknowledge the support received from the Australian Research Council. The second author is partially supported by the National Natural Science Foundation in China. He wishes to thank the University of New South Wales for its hospitality during the writing of the paper.

with the classical case, and also, the group W is no longer a Coxeter group. However, since the semi-simplicity of FC is simply equivalent to the condition that the order $|C|$ of the bottom C is non-zero in F , our strategy here is to find a q -analogue of the order of C , which is called the Poincaré polynomial of C , and with the invertibility of such a polynomial, to look for those idempotents e_λ . Thus, we eventually establish a q -analogue of the Morita equivalence (1) above. A by-product of our results is the introduction of the Poincaré polynomial d_W of the complex reflection group W . We shall see that the semi-simplicity of \mathbf{H} over a field F is equivalent to $d_W \neq 0$ in F .

We organize the paper as follows. In §1, we introduce the poset $\Lambda[m, r]$, which is isomorphic to $\Lambda(m, r)$ and discuss some combinatorics related to symmetric groups. In §2, a useful lemma (2.8) related to the poset structure on $\Lambda[m, r]$ is proved. Candidates of those idempotents e_λ are constructed in §3. The main results are presented in §4, where we prove that the invertibility of the ‘Poincaré polynomial’ $f_{m,r}$ is a necessary and sufficient condition for the existence of those idempotents e_λ , and the Morita equivalence is established. Finally, in §5, we lift the Morita equivalence to the endomorphism algebra level. Two by-products for \mathbf{H}_F over a field F in which $f_{m,r}$ is nonzero are the classification of simple modules and the criterion of semisimplicity.

The main results of the paper have been announced by the first author at the ‘Symposium on Modular representations of finite groups’, Charlottesville, Virginia, May 1998, and at the ‘International Conference on Representation Theory,’ Shanghai, June-July, 1998. At the Virginia conference, R. Dipper announced some Morita theorems for Ariki-Koike algebras joint with A. Mathas with quite different treatment. For example, our method works over the set $\Lambda(m, r)$ of compositions with m parts, while, in their method, they first treat the case where compositions have 2 parts. Thus, they announced a Morita equivalence between an Ariki-Koike algebra and a tensor product of two smaller such algebras. To obtain our result, they have to break two-part compositions further down.

Throughout, R denotes a commutative ring with identity 1.

1. The poset $\Lambda[m, r]$. Let r be a non-negative integer. A *composition* λ of r with $m > 0$ parts is a sequence $(\lambda_1, \dots, \lambda_m)$ of nonnegative integers such that $|\lambda| = \sum_{i=1}^m \lambda_i = r$, and λ is called a *partition* if the sequence is weakly decreasing. Let $\Lambda(m, r)$ (resp. $\Lambda(m, r)^+$) be the set of compositions (resp. partitions) of r with m -parts. With the usual dominance order \leq , both $\Lambda(m, r)$ and $\Lambda(m, r)^+$ are posets.

For notational convenience, we shall use another poset $\Lambda[m, r]$. For any $\lambda \in \Lambda(m, r)$, let $[\lambda] = [a_0, a_1, \dots, a_m]$ where $a_0 = 0$ and $a_i = \lambda_1 + \dots + \lambda_i$ for all i and put $\Lambda[m, r] = \{[\lambda] : \lambda \in \Lambda(m, r)\}$. The following results are almost obvious.

(1.1) Lemma. (a) *Alternatively, we have*

$$\Lambda[m, r] = \{[a_0, a_1, \dots, a_m] : 0 = a_0 \leq a_1 \leq \dots \leq a_m = r, a_i \in \mathbb{Z}, \forall i\}.$$

(b) *For any $\mathbf{a}, \mathbf{b} \in \Lambda[m, r]$, define $\mathbf{a} \preceq \mathbf{b}$ by setting $a_i \leq b_i$ for every i with $1 \leq i \leq m$. Then $\Lambda[m, r]$ is a poset with partial ordering \preceq . Moreover, the map $\Xi : \Lambda(m, r) \rightarrow \Lambda[m, r]$ defined by $\Xi(\lambda) = [\lambda]$ is an isomorphism between posets*

$(\Lambda(m, r), \trianglelefteq)$ and $(\Lambda[m, r], \preceq)$. In particular, we have for all $\lambda, \mu \in \Lambda(m, r)$

$$\lambda \trianglelefteq \mu \text{ if and only if } [\lambda] \preceq [\mu].$$

(c) If $\Theta = \Xi^{-1}$ is the inverse map of Ξ , then, for $\mathbf{a} = [a_i] \in \Lambda[m, r]$,

$$\Theta(\mathbf{a}) = (a_1 - a_0, a_2 - a_1, \dots, a_m - a_{m-1}).$$

If $\mathbf{a} \preceq \mathbf{b}$ and $\mathbf{a} \neq \mathbf{b}$, we write $\mathbf{a} \prec \mathbf{b}$.

(1.2) Notation. For any $\mathbf{a} = [a_0, \dots, a_m] \in \Lambda[m, r]$, let i (resp. j) be the minimal index such that $a_i \neq 0$ (resp. $a_j = r$) and define

- (a) $\mathbf{a}' = [0, r - a_{m-1}, \dots, r - a_1, r]$;
- (b) $\mathbf{a}_+ = [0, \dots, 0, a_i - 1, \dots, a_m - 1]$;
- (c) $\mathbf{a}_- = [a_0, a_1, \dots, a_{j-1}, r - 1, \dots, r - 1]$.

The notations we choose here are symmetric: if $\lambda = \Theta(\mathbf{a})$, then i (resp. j) is the minimal (resp. maximal) index with $\lambda_i \neq 0$ (resp. $\lambda_j \neq 0$), and we have $\mathbf{a}' = [(\lambda_m, \dots, \lambda_1)] = [\lambda^\circ]$, $\mathbf{a}_+ = [(0, \dots, 0, \lambda_i - 1, \lambda_{i+1}, \dots, \lambda_m)] = [\lambda_+]$, and $\mathbf{a}_- = [(\lambda_1, \dots, \lambda_{j-1}, \lambda_j - 1, 0, \dots, 0)] = [\lambda_-]$. Since $(\lambda_+)^{\circ} = (\lambda^{\circ})_-$ and $(\lambda_-)^{\circ} = (\lambda^{\circ})_+$, it follows that

- (d) $(\mathbf{a}_+)' = (\mathbf{a}')_-$ and $(\mathbf{a}_-)' = (\mathbf{a}')_+$.

Let $\mathfrak{S}_r = \mathfrak{S}_{\{1, \dots, r\}}$ be the symmetric group on r letters as in the introduction. Each element $w \in \mathfrak{S}_r$ can be expressed as a product of $s_{i_1} \cdots s_{i_k}$, where $s_i = (i, i+1)$ are basic transpositions. If k is minimal, such an expression is called a reduced expression of w . The number k is defined to be the length $l(w)$ of w . It is independent of the reduced expression of w and $l(w) = \#\{(i, j) \mid i < j, (i)w > (j)w\}$. For $\mathbf{a} = [a_i] \in \Lambda[m, r]$ with $\lambda = \Theta(\mathbf{a})$, let

$$\mathfrak{S}_{\mathbf{a}} = \mathfrak{S}_{\lambda} = \mathfrak{S}_{\{1, \dots, a_1\}} \times \mathfrak{S}_{\{a_1+1, \dots, a_2\}} \times \cdots \times \mathfrak{S}_{\{a_{m-1}+1, \dots, a_m\}}.$$

be the Young subgroup of \mathfrak{S}_r corresponding to \mathbf{a} , and $\mathcal{D}_{\mathbf{a}} = \mathcal{D}_{\lambda}$ the set of distinguished representatives of right \mathfrak{S}_{λ} -cosets. Similarly, for $\mathbf{a}, \mathbf{b} \in \Lambda[m, r]$, $\mathcal{D}_{\mathbf{a}, \mathbf{b}} = \mathcal{D}_{\mathbf{a}} \cap \mathcal{D}_{\mathbf{b}}^{-1}$ denotes the set of distinguished representatives of $\mathfrak{S}_{\mathbf{a}}\mathfrak{S}_{\mathbf{b}}$ double cosets.

For positive integers i, j , let $s_{i,i} = 1$ and

$$s_{i,j} = \begin{cases} s_{i-1} \cdots s_j, & \text{if } i > j, \\ s_i \cdots s_{j-1}, & \text{if } i < j. \end{cases}$$

(1.3) Lemma. (a) We have $\mathfrak{S}_r = \cup_{i=1}^r s_{i,r} \mathfrak{S}_{r-1}$, where $s_{i,r}$, $1 \leq i \leq r$, are distinguished \mathfrak{S}_{r-1} -coset representatives in \mathfrak{S}_r .

(b) Let $\mathbf{a} \in \Lambda[m, r]$ and $d \in \mathcal{D}_{\mathbf{a}, \mathbf{a}'}$. Then $d = s_{a_j, r} d_1$ for some j with $a_{j-1} < a_j$ and $d_1 \in \mathcal{D}_{\mathbf{b}, (\mathbf{a}_-)'}$, where $\mathbf{b} = [a_0, \dots, a_{j-1}, a_j - 1, a_{j+1} - 1, \dots, a_m - 1]$.

Proof. The statement (a) is well-known. Since $d \in \mathcal{D}_{\mathbf{a}'}^{-1}$, we have $d_1 \in \mathcal{D}_{\mathbf{a}'}^{-1} \cap \mathfrak{S}_{r-1}$. Therefore, $d_1 \in \mathcal{D}_{(\mathbf{a}')_+}^{-1} = \mathcal{D}_{(\mathbf{a}_-)'}$ by (1.2)(d). On the other hand, $d \in \mathcal{D}_{\mathbf{a}}$ implies that $i = a_j$ for some $1 \leq j \leq m$. Take the minimal j with $a_j = i$. Then $a_{j-1} \neq a_j$, and $\mathbf{b} \in \Lambda[m, r-1]$. If $d_1 \notin \mathcal{D}_{\mathbf{b}}$, then there is a $s_k \in \mathfrak{S}_{\mathbf{b}}$ such that $d_1 = s_k d_2$ with

$l(d_1) = l(d_2) + 1$, and $d = s_{a_j, r} s_k d_2$. If $k \geq a_j$, then $d = s_{k+1} s_{a_j, r} d_2$, contrary to $d \in \mathcal{D}_a$. If $k < a_j$, then $k \leq a_j - 2$ since $s_k \in \mathfrak{S}_b$. Thus, $d = s_k s_{a_j, r} d_2$, a contradiction again. Therefore, $d_1 \in \mathcal{D}_b$, and $d_1 \in \mathcal{D}_{b, (a_+)'}$. \square

Let $w_{i,j} = s_{i+1,1} s_{i+2,2} \cdots s_{i+j,j}$. Then $w_{i,j}$ is the following permutation

$$(1.4) \quad w_{i,j} = \begin{pmatrix} 1 & \cdots & i & i+1 & \cdots & i+j \\ j+1 & \cdots & j+i & 1 & \cdots & j \end{pmatrix}.$$

Let k be a non-negative integer. Define the k -shifted elements $s_{i,j}^{(k)}, w_{i,j}^{(k)}$ by setting $s_{i,j}^{(k)} = s_{i+k, j+k}$ and $w_{i,j}^{(k)} = s_{i+1,1}^{(k)} s_{i+2,2}^{(k)} \cdots s_{i+j,j}^{(k)}$, and $w_{i,j} = w_{i,j}^{(k)} = 1$, if $i = 0$ or $j = 0$. Note that $w_{i,j}^{(k)}$ is a permutation on $\{k+1, \dots, k+i+j\}$, and explicitly,

$$(1.5) \quad w_{i,j}^{(k)} = \begin{pmatrix} k+1 & \cdots & k+i & k+i+1 & \cdots & k+i+j \\ k+j+1 & \cdots & k+j+i & k+1 & \cdots & k+j \end{pmatrix}.$$

Obviously, we have $(s_{i,j}^{(k)})^{-1} = s_{j,i}^{(k)}$ and $(w_{i,j}^{(k)})^{-1} = w_{j,i}^{(k)}$.

For $\mathbf{a} = [a_i] \in \Lambda[m, r]$ with $\lambda = \Theta(\mathbf{a})$, let $w_{\mathbf{a}} \in \mathfrak{S}_r$ be defined by

$$(1.6) \quad (a_{i-1} + l)w_{\mathbf{a}} = r - a_i + l \text{ for all } i \text{ with } a_{i-1} < a_i, 1 \leq l \leq a_i - a_{i-1}.$$

In particular, we have $(a_i)w_{\mathbf{a}} = r - a_{i-1}$ if $a_{i-1} < a_i$. For example, for $\mathbf{a} = [0, i, i+j]$, $w_{\mathbf{a}} = w_{i,j}$, and for $\mathbf{b} = [0, 2, 5, 9]$,

$$w_{\mathbf{b}} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{8} & \underline{9} & \underline{5} & \underline{6} & \underline{7} & \underline{1} & \underline{2} & \underline{3} & \underline{4} \end{pmatrix}.$$

(1.7) Lemma. For $\mathbf{a} = [a_i] \in \Lambda[m, r]$ with $\lambda = \Theta(\mathbf{a})$,

$$\begin{aligned} w_{\mathbf{a}} &= w_{a_{m-1}, \lambda_m}^{(0)} w_{a_{m-2}, \lambda_{m-1}}^{(\lambda_m)} w_{a_{m-3}, \lambda_{m-2}}^{(\lambda_m + \lambda_{m-1})} \cdots w_{a_1, \lambda_2}^{(\lambda_m + \cdots + \lambda_3)} \\ &= w_{a_{m-1}, a_m - a_{m-1}}^{(0)} w_{a_{m-2}, a_{m-1} - a_{m-2}}^{(a_m - a_{m-1})} \cdots w_{a_1, a_2 - a_1}^{(a_m - a_2)}. \end{aligned}$$

Proof. The result follows immediately from (1.4) and (1.5). \square

We list some properties for the elements $w_{\mathbf{a}}$. First, $w_{\mathbf{a}}$ is distinguished and turns $\mathfrak{S}_{\mathbf{a}}$ into $\mathfrak{S}_{\mathbf{a}'}$.

(1.8) Lemma. For any $\mathbf{a} = [a_i] \in \Lambda[m, r]$, we have

- (a) $w_{\mathbf{a}}^{-1} = w_{\mathbf{a}'}$ and $w_{\mathbf{a}} \in \mathcal{D}_{\mathbf{a}}$. Hence $w_{\mathbf{a}} \in \mathcal{D}_{\mathbf{a}, \mathbf{a}'}$.
- (b) $w_{\mathbf{a}}^{-1} \mathfrak{S}_{\mathbf{a}} w_{\mathbf{a}} = \mathfrak{S}_{\mathbf{a}'}$. In particular, $w_{\mathbf{a}}^{-1} s_j w_{\mathbf{a}} = s_{(j)w_{\mathbf{a}}}$ for all $j \geq 1$ with $j \neq a_i$.

Proof. The first assertion in (a) follows from definition. Consider the root system Φ of type A_{r-1} and its subsystem $\Phi_{\mathbf{a}}$ whose Coxeter graph is obtained by removing all a_i -th vertices from that of Φ . For any $s_i \in \mathfrak{S}_r$, there is a simple root $\alpha_i = e_i - e_{i+1}$ with $s_i = s_{\alpha_i}$ and $w(\alpha_i) = e_{(i)w^{-1}} - e_{(i+1)w^{-1}}$ (see, e.g., [Hum]). From (1.6), we have $(a_j)w_{\mathbf{a}} = r - a_{j-1}$ for $a_{j-1} < a_j$, and $(i+1)w_{\mathbf{a}} = (i)w_{\mathbf{a}} + 1$ if $i \neq a_j$ for all $1 \leq j \leq m$. Thus, $w_{\mathbf{a}}^{-1}(\alpha_i)$ is a simple root. Therefore, $w_{\mathbf{a}}^{-1}$ stabilizes the positive root system of $\Phi_{\mathbf{a}}$, and consequently, $w_{\mathbf{a}} \in \mathcal{D}_{\mathbf{a}}$ and hence $w_{\mathbf{a}} \in \mathcal{D}_{\mathbf{a}, \mathbf{a}'}$, proving

(a). Because $w_{\mathbf{a}}^{-1}s_iw_{\mathbf{a}} = s_{(i)w_{\mathbf{a}}} \in \mathfrak{S}_{\mathbf{a}'}$ is a basic transposition, $w_{\mathbf{a}}^{-1}\mathfrak{S}_{\mathbf{a}}w_{\mathbf{a}} \subseteq \mathfrak{S}_{\mathbf{a}'}$. Therefore, by (a), $w_{\mathbf{a}}^{-1}\mathfrak{S}_{\mathbf{a}}w_{\mathbf{a}} = \mathfrak{S}_{\mathbf{a}'}$, proving (b). \square

We now look at the relation between $w_{\mathbf{a}}$ and $w_{\mathbf{a}_{\downarrow}}$. For $\mathbf{a} = [a_i] \in \Lambda[m, r]$ with minimal index k such that $a_k = r$, we have (see (1.2(c))) $\mathbf{a}_{\downarrow} = [0, a_1, \dots, a_{k-1}, r-1, \dots, r-1] \in \Lambda[m, r-1]$. Let $\mathbf{a}_i \in \Lambda[m, r]$, $i = 1, \dots, m$, be defined by

$$(1.9) \quad \mathbf{a}_i = \mathbf{a}_{\downarrow} + \mathbf{1}_i, \text{ where } \mathbf{1}_i = [0, \underbrace{0, \dots, 0}_{i-1}, 1, \dots, 1] \in \Lambda[m, 1].$$

Then $\mathbf{a}_i \in \Lambda[m, r]$ with $\mathbf{a}_1 \succ \mathbf{a}_2 \succ \dots \succ \mathbf{a}_m$ and $\mathbf{a}_k = \mathbf{a}$.

(1.10) Lemma. Write $\mathbf{a}_{\downarrow} = [b_0, b_1, \dots, b_m] \in \Lambda[m, r-1]$. Then, for any i , $1 \leq i \leq m$, $w_{\mathbf{a}_i} = s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}}$ with $l(w_{\mathbf{a}_i}) = l(s_{b_i+1, r}) + l(w_{\mathbf{a}_{\downarrow}}) + l(s_{r, r-b_{i-1}})$.

Proof. We prove $(l)w_{\mathbf{a}_i} = (l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}}$ for every l with $1 \leq l \leq r$.

Assume that $l \leq b_i$. Then there is a $j \leq i$ with $b_{j-1} < l \leq b_j$. Write $l = b_{j-1} + l' \leq b_j$. Then $(l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = (l)w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = ((r-1) - b_j + l')s_{r, r-b_{i-1}}$ using (1.6). Because $r-1-b_j+l' \geq r-b_{i-1}$ for $j < i$ and $r-1+l'-b_i < r-b_{i-1}$, we have

$$(l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = \begin{cases} r - b_j + l' & \text{if } j < i \\ (r-1) - b_j + l' & \text{if } j = i. \end{cases}$$

Thus, $(l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = (l)w_{\mathbf{a}_i}$. If $l = b_i + 1$, then $(l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = r - b_{i-1} = (l)w_{\mathbf{a}_i}$. Assume $l > b_i + 1$. Then there is a j with $j \geq i$ and $b_j < l-1 \leq b_{j+1}$. Write $l-1 = b_j + l'$. Because $r-1-b_{j+1}+l' < r-b_{i-1}$, we have $(l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = (l-1)w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}} = r-1-b_{j+1}+l'$. In this case, $l = (b_j + 1) + l'$ and $(l)w_{\mathbf{a}_i} = r - (b_{j+1} + 1) + l' = (l)s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}}$. So $w_{\mathbf{a}_i} = s_{b_i+1, r}w_{\mathbf{a}_{\downarrow}}s_{r, r-b_{i-1}}$. The length formula is obviously. \square

The elements $w_{\mathbf{a}} \in \mathfrak{S}_r$ and $w_{\mathbf{a}_{\downarrow}} \in \mathfrak{S}_{r-1}$ are related as follows.

(1.11) Corollary. Let $\mathbf{a} = [a_i] \in \Lambda[m, r]$ and suppose that k is the minimal index with $a_k \neq 0$. Then,

- (a) $w_{\mathbf{a}} = s_{a_k, r}w_{\mathbf{a}_{\downarrow}}$ with $l(w_{\mathbf{a}}) = l(w_{\mathbf{a}_{\downarrow}}) + l(s_{a_k, r})$.
- (b) $w_{\mathbf{a}'} = w_{(\mathbf{a}_{\downarrow})'}s_{r, a_k}$ with $l(w_{\mathbf{a}'}) = l(w_{(\mathbf{a}_{\downarrow})'}) + l(s_{r, a_k})$.

Proof. It follows immediately from (1.2)(d) and (1.10), or from (1.8) and (1.3). \square

2. Some zero divisors. Let $W = W_m^r$ be the group defined in the introduction. Then W is the group with r generators $\{s_0, s_1, s_2, \dots, s_{r-1}\} = S$ and relations:

$$\begin{aligned} s_0^m &= s_i^2 = 1, \text{ for all } 1 \leq i \leq r-1, \\ s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \text{ for } 1 \leq i \leq r-2, \\ s_0 s_1 s_0 s_1 &= s_1 s_0 s_1 s_0 \\ s_i s_j &= s_j s_i, \text{ if } 0 \leq i \leq j-2 \leq r-3. \end{aligned}$$

Let $t_1 = s_0$ and $t_i = s_{i-1}t_{i-1}s_{i-1}$, $2 \leq i \leq r$. Then $t_i t_j = t_j t_i$, $t_i^m = 1$ for $1 \leq i, j \leq r$ and $\{t_i \mid 1 \leq i \leq r\}$ generates the bottom C . We will identify the

subgroup of W generated by $\{s_i \mid 1 \leq i \leq r-1\}$ with \mathfrak{S}_r . Note that the group W_m^r is the symmetric (resp. hyperoctahedral) group if $m = 1$ (resp. $m = 2$).

A deformation of the group algebra of W has been given recently by Ariki and Koike [AK]. Let R be a commutative ring with 1 and $q, q^{-1}, u_1, \dots, u_m \in R$. The Ariki-Koike algebra $\mathbf{H} = \mathbf{H}_R = \mathbf{H}_m^r$ associated to the group W is an associative algebra over R with generators $T_i := T_{s_i}$, $0 \leq i \leq r-1$, subject to the relations:

$$(2.1) \quad \begin{cases} (1) & T_0 T_1 T_0 T_1 = T_1 T_0 T_1 T_0, \\ (2) & T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad \text{for } 1 \leq i \leq r-2 \\ (3) & T_i T_j = T_j T_i, \quad \text{if } |i-j| \geq 2 \\ (4) & (T_i - q)(T_i + 1) = 0, \quad \text{if } i \neq 0 \\ (5) & (T_0 - u_1) \cdots (T_0 - u_m) = 0. \end{cases}$$

Let $L_1 = T_0$ and $L_i = q^{-1} T_{i-1} L_{i-1} T_{i-1}$, $2 \leq i \leq r$. Then $L_i L_j = L_j L_i$, $1 \leq i, j \leq r$. The elements L_i , $1 \leq i \leq r$, generate an abelian subalgebra of \mathbf{H} . Since $L_i^m \neq 1$ in general, this subalgebra contains a proper submodule of rank $|C|$. So it cannot serve as a “bottom” of \mathbf{H} . However, \mathbf{H} is R -free of rank $|W|$ with basis [AK]:

$$(2.2) \quad \{L_1^{c_1} \cdots L_r^{c_r} T_w \mid w \in \mathfrak{S}_r, \text{ and } 0 \leq c_i \leq m-1, \forall i\}.$$

Let \mathcal{H} be the subalgebra generated by T_i for all $i \geq 1$. For a Young subgroup $\mathfrak{S}_{\mathbf{a}}$, let $\mathcal{H}(\mathfrak{S}_{\mathbf{a}})$ be the corresponding subalgebra. By (2.2), we have $\mathbf{H} = \bigoplus_{\mathbf{c}} L^{\mathbf{c}} \mathcal{H}$, where $\mathbf{c} = (c_1, \dots, c_r)$ and $L^{\mathbf{c}} = L_1^{c_1} \cdots L_r^{c_r}$. Thus, for every such \mathbf{c} , we have a projection map

$$(2.3) \quad pr_{\mathbf{c}} : \mathbf{H} \rightarrow L^{\mathbf{c}} \mathcal{H}.$$

The isomorphism from \mathbf{H} to the algebra \mathbf{H}^{op} opposite to \mathbf{H} induces an anti-automorphism

$$(2.4) \quad \iota : \mathbf{H} \rightarrow \mathbf{H} \text{ such that } \iota(T_i) = T_i.$$

Clearly, $\iota(L_i) = L_i$. We need the following commutator relations.

(2.5) Proposition. *Let \mathbf{H} be the Ariki-Koike algebra over a commutative ring R . Then*

- (a) T_i commutes with L_j if $j \neq i, i+1$.
- (b) T_i commutes with $L_i L_{i+1}$ and $L_i + L_{i+1}$.
- (c) T_i commutes with $\prod_{j=1}^k (L_j - x)$ for all $x \in R$ and $i \neq k$.
- (d) $L_i^k = q^{-1} T_{i-1} L_{i-1}^k T_{i-1} + (1 - q^{-1}) \sum_{c=1}^{k-1} L_i^c L_{i-1}^{k-c} T_{i-1}$ if $1 < i \leq r$ and $k \geq 1$.

Proof. See [AK, (3.3)] for statements (a-c), and [MM, (3.6)] for (d). \square

Following Graham and Lehrer [GL, (5.4)] (or [DJM, (3.1)]), we define for $\mathbf{a} = [a_i] \in \Lambda[m, r]$,

$$(2.6) \quad \begin{cases} \pi_{\mathbf{a}} = \pi_{a_1}(u_2) \cdots \pi_{a_{m-1}}(u_m), & \tilde{\pi}_{\mathbf{a}} = \pi_{a_1}(u_{m-1}) \cdots \pi_{a_{m-1}}(u_1), \\ \text{where } \pi_0(x) = 1 \text{ and } \pi_a(x) = \prod_{j=1}^a (L_j - x), \forall a > 0, x \in R. \end{cases}$$

Note that $\pi_{\mathbf{a}} = \pi_{\mathbf{a}_-}$ if $a_{m-1} \neq r$ and $\tilde{\pi}_{\mathbf{a}'} = \tilde{\pi}_{(\mathbf{a}_-)'}$ if $a_1 \neq 0$. Also, if, for $x_1, \dots, x_{m-1} \in R$, we put $\pi(\mathbf{a}; x_1, \dots, x_{m-1}) = \pi_{a_1}(x_1) \cdots \pi_{a_{m-1}}(x_{m-1})$, then $\pi_{\mathbf{a}} = \pi(\mathbf{a}; u_2, \dots, u_m)$ and $\tilde{\pi}_{\mathbf{a}} = \pi(\mathbf{a}; u_{m-1}, \dots, u_1)$.

(2.7) Corollary. (a) For $\mathbf{a} \in \Lambda[m, r]$, $\pi_{\mathbf{a}}$ and $\tilde{\pi}_{\mathbf{a}}$ commute with any element in $\mathcal{H}(\mathfrak{S}_{\mathbf{a}})$. In particular, for any $x \in R$, $\pi_r(x)$ is in the centre of \mathbf{H} .

(b) Assume $a_{j-1} < i \leq a_j$ for some j . Then

$$\pi_{\mathbf{a}} T_{i,r} = T_{i,a_j}(L_{a_j} - u_{j+1}) T_{a_j,a_{j+1}} \cdots (L_{a_{m-1}} - u_m) T_{a_{m-1},a_m} \pi_{\mathbf{b}},$$

where $\mathbf{b} = [0, \dots, 0, a_k, \dots, a_{j-1}, a_j - 1, \dots, a_m - 1] \in \Lambda[m, r-1]$ and $T_{i,j} = T_{s_{i,j}}$.

Proof. The statement (a) follows from (2.5)(c). Noting that $T_{i,j} = T_i \cdots T_{j-1}$ for $i < j$, we obtain (b) immediately from (2.5)(a)-(c).

We now prove the following useful result.

(2.8) Lemma. For $\mathbf{a}, \mathbf{b} \in \Lambda[m, r]$, we have $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{b}'} = 0$ and $\tilde{\pi}_{\mathbf{a}} \mathcal{H} \pi_{\mathbf{b}'} = 0$ unless $\mathbf{a} \preccurlyeq \mathbf{b}$.

Proof. Using the anti-automorphism ι of \mathbf{H} in (2.4) and the fact $\mathbf{b}' \preccurlyeq \mathbf{a}'$ if and only if $\mathbf{a} \preccurlyeq \mathbf{b}$, we see that both assertions in (2.8) are equivalent. Therefore, we only need to prove $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{b}'} = 0$ unless $\mathbf{a} \preccurlyeq \mathbf{b}$.

We apply induction on r . Let $r = 1$. If $\mathbf{a} \not\preccurlyeq \mathbf{b}$, then there is an i with $a_i > b_i$. Since $b_i, a_i \leq 1$ for all i , we have $b_i = 0$ and $a_i = 1$. By (2.6), $\prod_{k=i}^{m-1} (L_1 - u_{k+1}) = \prod_{k=i+1}^m (L_1 - u_k)$ (resp. $\prod_{k=1}^i (L_1 - u_k)$) is a factor of $\pi_{\mathbf{a}}$ (resp. $\tilde{\pi}_{\mathbf{b}'}$). Therefore, $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{b}'} = 0$ by (2.1)(5), and (2.8) is true for $r = 1$. Assume $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{b}'} = 0$ for all $\mathbf{a}, \mathbf{b} \in \Lambda[m, r-1]$ with $\mathbf{a} \not\preccurlyeq \mathbf{b}$.

Let i and j be the minimal indices with $b_i \neq 0$ and $a_j \neq 0$, respectively. Because $b_k = 0$ for all $k < i$, $\pi_{b_m-b_k}(u_k) = \pi_r(u_k)$, which are in the centre of \mathbf{H} (see (2.7)(a)). Therefore, for any $w \in \mathfrak{S}_r$, $\pi_{\mathbf{a}} T_w \tilde{\pi}_{\mathbf{b}'}$ contains a factor $\prod_{k=j+1}^m (L_1 - u_k) \prod_{k=1}^{i-1} (L_1 - u_k)$. By (2.1)(5), $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{b}'} = 0$ unless $i \leq j$. On the other hand, take $w \in \mathfrak{S}_r$ with $\pi_{\mathbf{a}} T_w \tilde{\pi}_{\mathbf{b}'} \neq 0$. Write $w = dy$ with $y \in \mathfrak{S}_{r-1}$ and $d = s_{k,r}$ for some $1 \leq k \leq r$ (see (1.3)). By (2.7)(b), $\pi_{\mathbf{a}} T_d = h \pi_{\mathbf{a}_-}$ for some $h \in \mathbf{H}$. By (2.6), $\tilde{\pi}_{\mathbf{b}'} = \tilde{\pi}_{(\mathbf{b}_-)'}(L_r - u_{i-1}) \cdots (L_r - u_1)$. So, $\pi_{\mathbf{a}} T_w \tilde{\pi}_{\mathbf{b}'} \neq 0$ implies $\pi_{\mathbf{a}_-} T_y \tilde{\pi}_{(\mathbf{b}_-)'}$ is not zero. Now, by induction, $\mathbf{a}_- \preccurlyeq \mathbf{b}_-$, which implies $\mathbf{a} \preccurlyeq \mathbf{b}$ since $i \leq j$. \square

3. Idempotents. In this section, idempotents $e_{\mathbf{a}}$ for all $\mathbf{a} \in \Lambda[m, r]$ will be constructed under a certain condition.

(3.1) Proposition. For any $\mathbf{a} \in \Lambda[m, r]$, let $v_{\mathbf{a}} = \pi_{\mathbf{a}} T_{w_{\mathbf{a}}} \tilde{\pi}_{\mathbf{a}'}$. Then we have:

(a) $\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{a}'} = v_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) = \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) v_{\mathbf{a}}$. Moreover, $v_{\mathbf{a}} T_i = T_{(i)w_{\mathbf{a}}}^{-1} v_{\mathbf{a}}$ for any $s_i \in \mathfrak{S}_{\mathbf{a}'}$.

(b) $v_{\mathbf{a}} L_i \in v_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_i) \cap v_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$ and $L_i v_{\mathbf{a}} \in \mathcal{H}(\mathfrak{S}_i) v_{\mathbf{a}} \cap \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) v_{\mathbf{a}}$ for every $i = 1, \dots, r$. In particular, $v_{\mathbf{a}} L_i = u_j v_{\mathbf{a}}$ if $i = r - a_j + 1$ for some j with $a_{j-1} < a_j$.

(c) $\pi_{\mathbf{a}} \mathbf{H} \tilde{\pi}_{\mathbf{a}'} = \pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{a}'}$.

(d) $v_{\mathbf{a}} \mathbf{H} = v_{\mathbf{a}} \mathcal{H}$.

Proof. Let $d \in \mathcal{D}_{\mathbf{a}, \mathbf{a}'}$. We prove $\pi_{\mathbf{a}} T_d \tilde{\pi}_{\mathbf{a}'} = 0$ unless $d = w_{\mathbf{a}}$. Obviously, this result is true for $r = 1$, and assume that $r > 1$ and that the result is true for $r - 1$. By

(1.3), $d = s_{i,r}d_1$ with $d_1 \in \mathfrak{S}_{r-1}$ and $i \geq a_k$, where k is the minimal index with $a_k \neq 0$. If $i > a_k$, then there is an index j with $a_{j-1} < i \leq a_j$, $j > k$. By (2.7)(b),

$$\pi_{\mathbf{a}}T_d = T_{i,a_j}(L_{a_j} - u_{j+1})T_{a_j,a_{j+1}}(L_{a_{j+1}} - u_{j+2}) \cdots (L_{a_{m-1}} - u_m)T_{a_{m-1},a_m}\pi_{\mathbf{b}}T_{d_1},$$

where $\mathbf{b} = [0, \dots, 0, a_k, \dots, a_{j-1}, a_j - 1, \dots, a_m - 1] \in \Lambda[m, r-1]$ and $\mathbf{b} \not\leq \mathbf{a}_+$. By (2.6), $\tilde{\pi}_{\mathbf{a}'} = \tilde{\pi}_{(\mathbf{a}_+)'}(L_r - u_{k-1}) \cdots (L_r - u_1)$. We have $\pi_{\mathbf{b}}T_{d_1}\tilde{\pi}_{(\mathbf{a}_+)'}$ by (2.8). So $\pi_{\mathbf{a}}T_d\tilde{\pi}_{\mathbf{a}'} = 0$, a contradiction, proving $i = a_k$. From (1.3)(b), we have $d_1 \in \mathcal{D}_{\mathbf{a}_+, (\mathbf{a}_+)'}$. Because

$$\begin{aligned} \pi_{\mathbf{a}}T_d\tilde{\pi}_{\mathbf{a}'} &= (L_{a_k} - u_{j+1})T_{a_k,a_{k+1}}(L_{a_{k+1}} - u_{k+2}) \cdots (L_{a_{m-1}} - u_m)T_{a_{m-1},a_m} \\ &\quad \pi_{\mathbf{a}_+}T_{d_1}\tilde{\pi}_{(\mathbf{a}_+)'}(L_r - u_{k-1}) \cdots (L_r - u_1), \end{aligned}$$

we have $\pi_{\mathbf{a}_+}T_{d_1}\tilde{\pi}_{(\mathbf{a}_+)'}$ by (1.11). Now, the first assertion in (a) follows immediately from the $\mathfrak{S}_{\mathbf{a}}\text{-}\mathfrak{S}_{\mathbf{a}'}$ double coset decomposition of \mathfrak{S}_r .

For notational simplicity, we write

$$h_{\mathbf{a}} = T_{w_{\mathbf{a}}}.$$

By (1.8), we have $h_{\mathbf{a}}T_i = T_{(i)w_{\mathbf{a}}^{-1}}h_{\mathbf{a}}$ for every $s_i \in \mathfrak{S}_{\mathbf{a}'}$. Therefore, $v_{\mathbf{a}}T_i = \pi_{\mathbf{a}}h_{\mathbf{a}}T_i\tilde{\pi}_{\mathbf{a}'} = \pi_{\mathbf{a}}T_{(i)w_{\mathbf{a}}^{-1}}h_{\mathbf{a}}\tilde{\pi}_{\mathbf{a}'} = T_{(i)w_{\mathbf{a}}^{-1}}v_{\mathbf{a}}$, proving the second assertion in (a).

To see (b), we first treat the case $i = a_m - a_j + 1$ with j minimal. Then $a_j > a_{j-1}$, and $\mathbf{b} = [a_0, a_1, \dots, a_{j-1}, a_j - 1, a_{j+1}, \dots, a_m] \in \Lambda[m, r]$. Obviously, $\mathbf{b} \not\leq \mathbf{a}$. By (2.8), $\pi_{\mathbf{a}}h_{\mathbf{a}}\tilde{\pi}_{\mathbf{b}'} = 0$. By (2.6), we have $\tilde{\pi}_{\mathbf{a}'}L_i = \tilde{\pi}_{\mathbf{b}'} + u_j\tilde{\pi}_{\mathbf{a}'}$, where $\mathbf{b}' = [0, a_m - a_{m-1}, \dots, a_m - a_{j-1}, a_m - (a_j - 1), \dots, a_m - a_1, a_m]$. Thus, $v_{\mathbf{a}}L_i = \pi_{\mathbf{a}}h_{\mathbf{a}}\tilde{\pi}_{\mathbf{b}'} + u_jv_{\mathbf{a}} = u_jv_{\mathbf{a}}$. Now, assume $i \neq a_m - a_j + 1$, $1 \leq j \leq m$. Then there is a k with $a_m - a_k + 1 < i \leq a_m - a_{k-1}$. Write $i = (a_m - a_k + 1) + l$. Then $L_i = q^{-l}T_{i,a_m-a_k+1}L_{a_m-a_k+1}T_{a_m-a_k+1,i}$. By (a), $v_{\mathbf{a}}L_i = u_kq^{-l}v_{\mathbf{a}}T_{i,a_m-a_k+1}T_{a_m-a_k+1,i} \in v_{\mathbf{a}}\mathcal{H}(\mathfrak{S}_i) \cap v_{\mathbf{a}}\mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$. By a symmetric argument, one can prove $L_iv_{\mathbf{a}} \in \mathcal{H}(\mathfrak{S}_i)v_{\mathbf{a}} \cap \mathcal{H}(\mathfrak{S}_{\mathbf{a}})v_{\mathbf{a}}$.

By (b) and (a), $L_iv_{\mathbf{a}}\mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) \subseteq \mathcal{H}(\mathfrak{S}_{\mathbf{a}})v_{\mathbf{a}}\mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) = v_{\mathbf{a}}\mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$ for every $i = 1, \dots, r$. Thus, (c) follows immediately from (2.2).

For arbitrary $v_{\mathbf{a}}T_w \in v_{\mathbf{a}}\mathcal{H}$ with $w \in \mathfrak{S}_r$, write $w = dy$ with $y \in \mathfrak{S}_{\{2, \dots, r\}}$ and $d \in \mathfrak{S}_r/\mathfrak{S}_{\{2, \dots, r\}}$. Then $d = s_{i,1}$ with $1 \leq i \leq r$. By (b), $v_{\mathbf{a}}T_wT_0 = v_{\mathbf{a}}T_{i,1}T_0T_y \in v_{\mathbf{a}}\mathcal{H}$. Thus, $v_{\mathbf{a}}\mathcal{H}$ is stable under the right action of T_0 , and $v_{\mathbf{a}}\mathbf{H} = v_{\mathbf{a}}\mathcal{H}$. \square

Recall from (2.7) that $T_{i,j} = T_{s_{i,j}}$ for $i \neq 0 \neq j$. For $\mathbf{a} \in \Lambda[m, 0]$, let $v_{\mathbf{a}} = 1$.

(3.2) Lemma. *Let x_1, \dots, x_{m-1} be elements in the commutative ring R and $\mathbf{a} \in \Lambda[m, r]$ with $a_1 \neq 0$. Then there is an integer c depending on \mathbf{a} such that*

$$T_{a_m, a_{m-1}}(L_{a_{m-1}} - x_{m-1}) \cdots T_{a_2, a_1}(L_{a_1} - x_1) - q^c L_r^{m-1} T_{a_1, a_m}^{-1}$$

is in the free R -submodule spanned by $\{L_1^{c_1} \cdots L_r^{c_r} T_w \mid w \in \mathfrak{S}_r, c_j < m-1, \forall j\}$.

Proof. Let U_i ($1 \leq i \leq m-1$) be the free R -submodule spanned by

$$\{L_1^{c_1} \cdots L_r^{c_r} T_w \mid w \in \mathfrak{S}_{\{a_i, a_i+1, \dots, r\}}, c_j < m-i, \forall j\}.$$

We claim that, for any $i = 1, \dots, m-1$,

$$(3.3) \quad T_{a_m, a_{m-1}}(L_{a_{m-1}} - x_{m-1}) \cdots T_{a_{i+1}, a_i}(L_{a_i} - x_i) - q^c L_r^{m-i} T_{a_i, a_m}^{-1} \in U_i.$$

Apply induction on i . The result for $i = m-1$ is true since $T_{a_m, a_{m-1}}(L_{a_{m-1}} - x_{m-1}) = q^{a_m - a_{m-1}} L_{a_m} T_{a_{m-1}, a_m}^{-1} - x_{m-1} T_{a_m, a_{m-1}}$. For $i < m-1$, we have, by induction,

$$\begin{aligned} & T_{a_m, a_{m-1}}(L_{a_{m-1}} - x_{m-1}) \cdots T_{a_{i+1}, a_i}(L_{a_i} - x_i) \\ &= (q^c L_r^{m-i-1} T_{a_{i+1}, a_m}^{-1} + h) T_{a_{i+1}, a_i}(L_{a_i} - x_i) \\ &= q^c L_r^{m-i-1} T_{a_{i+1}, a_m}^{-1} T_{a_{i+1}, a_i}(L_{a_i} - x_i) + h T_{a_{i+1}, a_i}(L_{a_i} - x_i) \end{aligned}$$

for some $h \in U_{i+1}$. We may assume $h = L_1^{c_1} \cdots L_r^{c_r} T_w$ with $w \in \mathfrak{S}_{\{a_{i+1}, \dots, r\}}$ and $c_j < m - (i+1)$, $\forall j$, without loss of generality. Write $w = s_{k, a_{i+1}} y$ with $y \in \mathfrak{S}_{\{a_{i+1}+1, \dots, r\}}$ (cf. (1.3)). Then $T_w T_{a_{i+1}, a_i}(L_{a_i} - x_{i+1}) = q^{k-a_i} L_k T_{a_i, k}^{-1} T_y - x_{i+1} T_{k, a_i} T_y$. Thus, we have $h T_{a_{i+1}, a_i}(L_{a_i} - x_{i+1}) \in U_i$. Noting $q T_i^{-1} = T_i - (q-1)$ and $L_r^{m-i-1} T_{r, a_{i+1}} T_{a_{i+1}, a_i} L_{a_i} = q^{r-a_i} L_r^{m-i} T_{a_i, r}^{-1}$, we may write

$$L_r^{m-i-1} T_{a_{i+1}, a_m}^{-1} T_{a_{i+1}, a_i}(L_{a_i} - x_{i+1}) = q^{c'} L_r^{m-i} T_{a_i, r}^{-1} + h'$$

for some $h' \in U_i$ and $c' \in \mathbb{Z}$. Therefore, (3.3) holds for i , proving the claim. Now, the required result follows from the case $i = 1$. \square

(3.4) Theorem. *Let \mathbf{H} be the Ariki-Koike algebra over R . For any $\mathbf{a} \in \Lambda[m, r]$, $v_{\mathbf{a}} \mathbf{H}$ is a free R -module with basis $\{v_{\mathbf{a}} T_w \mid w \in \mathfrak{S}_r\}$.*

Proof. For $h = (m-1, \dots, m-1)$, let pr_h be the projection defined in (2.3). First, we prove

$$(3.5) \quad pr_h(v_{\mathbf{a}}) = q^c \prod_{j=1}^r L_j^{m-1} h_{\mathbf{a}'}^{-1}$$

for some integer $c \in \mathbb{Z}$ depending on \mathbf{a} . We prove (3.5) by induction on r . Obviously, the result is true for $r = 1$. Assume now that $r > 1$ and that (3.5) holds for $r-1$. For $\mathbf{a} \in \Lambda[m, r]$ with minimal index k , $a_k = r$, recall

$$\begin{aligned} \mathbf{a}' &= [0, 0, \dots, 0, r - a_{k-1}, \dots, r - a_1, r], \\ \mathbf{a}_{\rightarrow} &= [0, a_1, \dots, a_{k-1}, r-1, \dots, r-1], \\ (\mathbf{a}_{\rightarrow})' &= [0, \dots, 0, r-1 - a_{k-1}, \dots, r-1 - a_1, r-1], \text{ and} \\ \pi_{\mathbf{a}} &= \pi_{\mathbf{a}_{\rightarrow}}(L_r - u_{k+1}) \cdots (L_r - u_m). \end{aligned}$$

Then, noting from (1.10) that $w_{\mathbf{a}} = w_{\mathbf{a}_k} = w_{\mathbf{a}_{\rightarrow}} s_{r, r-a_{k-1}}$, we have

$$\begin{aligned} v_{\mathbf{a}} &= v_{\mathbf{a}_{\rightarrow}}(L_r - u_{k+1}) \cdots (L_r - u_m) \\ &= T_{r, r-a_1}(L_{r-a_1} - u_1) \cdots T_{r-a_{k-2}, r-a_{k-1}}(L_{r-a_{k-1}} - u_{k-1}). \end{aligned}$$

Thus, applying (3.2) to $[0, r - a_{k-1}, \dots, r - a_1, r, \dots, r]$, we have

$$v_{\mathbf{a}} = v_{\mathbf{a}_{\rightarrow}}(q^{c_2} L_r^{m-1} T_{r-a_{k-1}, r}^{-1} + h_1)$$

where $c_2 \in \mathbb{Z}$, $h_1 \in U_1$ (see the proof of (3.2)). On the other hand, by induction, we have, for some integer c_1 , $v_{\mathbf{a}_{\rightarrow}} = q^{c_1} \prod_{j=1}^{r-1} L_j^{m-1} h_{(\mathbf{a}_{\rightarrow})'}^{-1} + h$, where h is a linear combinations of elements $L_1^{d_1} \cdots L_{r-1}^{d_{r-1}} T_w$ with $w \in \mathfrak{S}_{r-1}$. Since L_r commutes with any elements in \mathbf{H}_m^{r-1} , we have $pr_h(q^{c_1} \prod_{j=1}^{r-1} L_j^{m-1} h_{(\mathbf{a}_{\rightarrow})'}^{-1} h_1) = 0$, $pr_h(hq^{c_2} L_r^{m-1} T_{r-a_k}^{-1}) = 0$ and $pr_h(hh_1) = 0$. Therefore, (3.5) holds (cf. (1.11)(a)).

Now we are ready to prove that the set $\{v_{\mathbf{a}} T_w \mid w \in \mathfrak{S}_r\}$ is linearly independent. Suppose $\sum_{w \in \mathfrak{S}_r} c_w v_{\mathbf{a}} T_w = 0$. Then, by (3.5), there is a $c \in \mathbb{Z}$ depending on \mathbf{a} such that

$$pr_h\left(\sum_{w \in \mathfrak{S}_r} c_w v_{\mathbf{a}} T_w\right) = \sum_{w \in \mathfrak{S}_r} c_w q^c L_1^{m-1} \cdots L_r^{m-1} h_{\mathbf{a}'}^{-1} T_w = 0.$$

Because $h_{\mathbf{a}'}^{-1}$ and q are invertible, we have $c_w = 0$ for every $w \in \mathfrak{S}_r$ by (2.2). Therefore, $\{v_{\mathbf{a}} T_w \mid w \in \mathfrak{S}_r\}$ is a linearly independent set. By (3.1)(d), $v_{\mathbf{a}} \mathbf{H} = v_{\mathbf{a}} \mathcal{H}$. Thus, $v_{\mathbf{a}} \mathbf{H}$ is a free R -module with basis $\{v_{\mathbf{a}} T_w \mid w \in \mathfrak{S}_r\}$. \square

(3.6) Proposition. *Let $\mathbf{a} \in \Lambda[m, r]$ and write $v_{\mathbf{a}} h_{\mathbf{a}'} v_{\mathbf{a}} = v_{\mathbf{a}} z_{\mathbf{a}'} = z_{\mathbf{a}} v_{\mathbf{a}}$, where $z_{\mathbf{a}} \in \mathcal{H}(\mathfrak{S}_{\mathbf{a}})$ and $z_{\mathbf{a}'} \in \mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$. Then $z_{\mathbf{a}}$ (resp. $z_{\mathbf{a}'}$) is in the center of the algebra $\mathcal{H}(\mathfrak{S}_{\mathbf{a}})$ (resp. $\mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$).*

Proof. The existence of $z_{\mathbf{a}}$ and $z_{\mathbf{a}'}$ follows from (3.1a,c). We need only to prove that $z_{\mathbf{a}'}$ is in the center of $\mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$. One may prove that $z_{\mathbf{a}}$ is in the center of $\mathcal{H}(\mathfrak{S}_{\mathbf{a}})$, similarly.

Let $T_i \in \mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$. Then $i \neq a_m - a_j$ for $1 \leq j \leq m$. By (3.1)(a), $v_{\mathbf{a}} h_{\mathbf{a}'} v_{\mathbf{a}} T_i = v_{\mathbf{a}} h_{\mathbf{a}'} T_{(i)w_{\mathbf{a}}^{-1}} v_{\mathbf{a}} = v_{\mathbf{a}} T_i h_{\mathbf{a}'} v_{\mathbf{a}} = T_{(i)w_{\mathbf{a}}^{-1}} v_{\mathbf{a}} z_{\mathbf{a}'} = v_{\mathbf{a}} T_i z_{\mathbf{a}'}$. Therefore, $v_{\mathbf{a}} z_{\mathbf{a}'} T_i = v_{\mathbf{a}} T_i z_{\mathbf{a}'}$. By (3.4), $z_{\mathbf{a}'} T_i = T_i z_{\mathbf{a}'}$. So, $z_{\mathbf{a}'}$ is in the center of $\mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$. \square

(3.7) Corollary. *We have $T_{w_{\mathbf{a}}} z_{\mathbf{a}'} = z_{\mathbf{a}} T_{w_{\mathbf{a}}}$.*

Proof. By (1.8), we may write $T_{w_{\mathbf{a}}} z_{\mathbf{a}'} = z T_{w_{\mathbf{a}}}$ for some $z \in \mathcal{H}$. Then $v_{\mathbf{a}} z_{\mathbf{a}'} = z v_{\mathbf{a}}$, and hence $z = z_{\mathbf{a}}$ by (3.4). \square

(3.8) Corollary. *For $\mathbf{a} \in \Lambda[m, r]$, $v_{\mathbf{a}} \mathbf{H} v_{\mathbf{a}} = v_{\mathbf{a}} z_{\mathbf{a}'} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) = \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) z_{\mathbf{a}} v_{\mathbf{a}}$.*

Proof. Let ι be the anti-automorphism of \mathbf{H} defined in (2.4). By (3.1a), we have $\tilde{\pi}_{\mathbf{a}'} \mathcal{H} \pi_{\mathbf{a}} = \iota(\pi_{\mathbf{a}} \mathcal{H} \tilde{\pi}_{\mathbf{a}'}) = \iota(\mathcal{H}(\mathfrak{S}_{\mathbf{a}}) v_{\mathbf{a}}) = \tilde{\pi}_{\mathbf{a}'} h_{\mathbf{a}'} \pi_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}})$. So,

$$\begin{aligned} v_{\mathbf{a}} \mathbf{H} v_{\mathbf{a}} &= v_{\mathbf{a}} \mathcal{H} v_{\mathbf{a}} = \pi_{\mathbf{a}} h_{\mathbf{a}} (\tilde{\pi}_{\mathbf{a}'} \mathcal{H} \pi_{\mathbf{a}}) h_{\mathbf{a}} \tilde{\pi}_{\mathbf{a}'} \\ &= \pi_{\mathbf{a}} h_{\mathbf{a}} \tilde{\pi}_{\mathbf{a}'} h_{\mathbf{a}'} \pi_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) h_{\mathbf{a}} \tilde{\pi}_{\mathbf{a}'} \\ &= v_{\mathbf{a}} h_{\mathbf{a}'} v_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) = v_{\mathbf{a}} z_{\mathbf{a}'} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'}). \quad \square \end{aligned}$$

(3.9) Proposition. *Let $\mathbf{a} \in \Lambda[m, r]$. Then the following are equivalent:*

(a) $z_{\mathbf{a}}$ (equivalently $z_{\mathbf{a}'}$) is invertible.

- (b) $e_{\mathbf{a}} = v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1}$ is an idempotent.
(c) $v_{\mathbf{a}} \mathbf{H}$ is a projective right \mathbf{H} -module.

Proof. Let $J = v_{\mathbf{a}} \mathbf{H}$. By (3.1)(d), we have $J^2 = v_{\mathbf{a}} z_{\mathbf{a}'} \mathcal{H}$. By (3.4), the map from \mathcal{H} to $v_{\mathbf{a}} \mathcal{H}$ sending $h \rightarrow v_{\mathbf{a}} h$ for $h \in \mathcal{H}$ is injective. Therefore, $J^2 = J$ if and only if $z_{\mathbf{a}'} \mathcal{H} = \mathcal{H}$, which is equivalent to say that $z_{\mathbf{a}'}$ is invertible.

If J is projective, then $J = e \mathbf{H}$ for some idempotent $e \in \mathbf{H}$, and $J = J^2$. So, $z_{\mathbf{a}'}$, and hence $z_{\mathbf{a}}$, is invertible. Now, assume $z_{\mathbf{a}}$ is invertible. Putting $e_{\mathbf{a}} = v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1}$ and recalling $h_{\mathbf{a}'} = T_{w_{\mathbf{a}'}}$, we have

$$\begin{aligned} e_{\mathbf{a}}^2 &= v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} = v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} \pi_{\mathbf{a}} h_{\mathbf{a}} \tilde{\pi}_{\mathbf{a}'} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} \\ &= v_{\mathbf{a}} h_{\mathbf{a}'} \pi_{\mathbf{a}} z_{\mathbf{a}}^{-1} h_{\mathbf{a}} \tilde{\pi}_{\mathbf{a}'} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} \\ &= v_{\mathbf{a}} h_{\mathbf{a}'} v_{\mathbf{a}} z_{\mathbf{a}'}^{-1} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} = v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} = e_{\mathbf{a}}. \end{aligned}$$

Thus, $e_{\mathbf{a}}$ is an idempotent and $J = v_{\mathbf{a}} \mathbf{H} = e_{\mathbf{a}} \mathbf{H}$. Therefore, $v_{\mathbf{a}} \mathbf{H}$ is projective. \square

(3.10) Proposition. *Let $\mathbf{a}, \mathbf{b} \in \Lambda[m, r]$ and assume that $z_{\mathbf{a}}, z_{\mathbf{b}}$ are invertible. Then $e_{\mathbf{a}} \mathbf{H} e_{\mathbf{b}} = 0$ unless $\mathbf{a} = \mathbf{b}$. Moreover, we have $e_{\mathbf{a}} \mathbf{H} e_{\mathbf{a}} \cong \mathcal{H}(\mathfrak{S}_{\mathbf{a}})$.*

Proof. By (2.8) and (3.1)(c), $v_{\mathbf{a}} \mathbf{H} v_{\mathbf{b}} \neq 0$ implies $\mathbf{a} \preccurlyeq \mathbf{b}$ and $\mathbf{a}' \preccurlyeq \mathbf{b}'$. So $\mathbf{a} = \mathbf{b}$. The second assertion follows from the following equality:

$$\begin{aligned} e_{\mathbf{a}} \mathbf{H} e_{\mathbf{a}} &= v_{\mathbf{a}} \mathbf{H} v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} = v_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}'}) h_{\mathbf{a}'} \text{ by (3.8)} \\ &= v_{\mathbf{a}} h_{\mathbf{a}'} \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) \cong \mathcal{H}(\mathfrak{S}_{\mathbf{a}}). \quad \square \end{aligned}$$

(3.11) Corollary. *Assume that $z_{\mathbf{a}}$ are invertible for all $\mathbf{a} \in \Lambda[m, r]$ and let $\varepsilon = \sum_{\mathbf{a} \in \Lambda[m, r]} e_{\mathbf{a}}$. Then $\varepsilon \mathbf{H} \varepsilon \cong \oplus_{\mathbf{a} \in \Lambda[m, r]} \mathcal{H}(\mathfrak{S}_{\mathbf{a}})$.*

Proof. The result follows from (3.10). \square

4. The Poincaré polynomial of W . To generalize the Morita equivalence (1) given in the introduction, we need two more ingredients: First, we want to know when the hypothesis in (3.11) holds. This leads to the introduction of the Poincaré polynomial of the bottom. Second, we need to prove that the direct sum $\oplus_{\mathbf{a} \in \Lambda[m, r]} v_{\mathbf{a}} \mathbf{H}$ is a projective generator for the category $\mathbf{H}\text{-mod}$. The latter requires that R is an integral domain.

(4.1) Lemma. *For $\mathbf{r}_i = (0, \underbrace{0 \cdots 0}_{i-1}, r \cdots, r) \in \Lambda[m, r]$, let $z_{\mathbf{r}'_i} \in \mathfrak{S}_{\mathbf{r}'_i}$ defined in*

$$(3.6). \text{ Then } z_{\mathbf{r}'_i} = \prod_{\substack{1 \leq j \leq m \\ j \neq i}} \prod_{k=1}^r (u_i q^{1-k} T_{k,1} T_{1,k} - u_j).$$

Proof. Since $w_{\mathbf{r}_i} = 1$, we have $h_{\mathbf{r}_i} = T_{w_{\mathbf{r}_i}} = 1$. It follows from (2.1)(5) and (2.7)(a) that

$$v_{\mathbf{r}_i}^2 = v_{\mathbf{r}_i} \prod_{\substack{1 \leq j \leq m \\ j \neq i}} \prod_{k=1}^r (u_i q^{1-k} T_{k,1} T_{1,k} - u_j).$$

On the other hand, by (3.6), we have $v_{\mathbf{r}_i}^2 = v_{\mathbf{r}_i} z_{\mathbf{r}'_i}$. Now, the result follows from (3.4) immediately. \square

(4.2) Definition. For positive integers m, r and $i = 1, \dots, m$, let

$$f_{m,r,i} = f_{m,r,i}(u_1, \dots, u_m, q) = \prod_{\substack{1 \leq j \leq m \\ i \neq j}} \prod_{k=1-r}^{r-1} (u_j q^k - u_i),$$

$$f_{m,r} = f_{m,r}(u_1, \dots, u_m, q) = \prod_{i=1}^{m-1} \prod_{j=i+1}^m \prod_{k=1-r}^{r-1} (u_j q^k - u_i).$$

We shall see below that the polynomial $f_{m,r}$ can be viewed as the Poincaré polynomial of the bottom C . Let $d_{\mathfrak{S}_r}$ be the Poincaré polynomial of \mathfrak{S}_r , i.e., $d_{\mathfrak{S}_r} = \sum_{w \in \mathfrak{S}_r} q^{l(w)}$, then the polynomial $d_W = f_{m,r} d_{\mathfrak{S}_r}$ is called the *Poincaré polynomial* of the complex reflection group W .

(4.3) Proposition. *Maintain the notation introduced above. The element $z_{\mathbf{r}'_i}$ is invertible if and only if $f_{m,r,i}$ is invertible in R .*

Proof. By (4.1), we see that the invertibility of $z_{\mathbf{r}'_i}$ is equivalent to the invertibility of $h_{ij} = \prod_{k=1}^r (u_i q^{1-k} T_{k,1} T_{1,k} - u_j)$ for all $j \neq i$. By [DJ2, (4.3)], we see that, if R is a field, then h_{ij} is invertible if and only if $f_{m,r,i}$ is invertible. The general case follows by an argument similar to the one for [DJ2, (4.5)]. \square

We need some preparation in order to get the main result of this paper. As before, we assume that R is a commutative ring (with 1). For any $\mathbf{a} \in \Lambda[m, r]$ and $x_1, \dots, x_{m-1} \in R$, recall the element defined after (2.6)

$$\pi(\mathbf{a}) = \pi(\mathbf{a}; x_1, \dots, x_{m-1}) = \pi_{a_1}(x_1) \cdots \pi_{a_{m-1}}(x_{m-1}).$$

Write $\pi(\mathbf{a})$ as a polynomial in L_i . The degree of this polynomial is denoted $\deg_i(\pi(\mathbf{a}))$.

(4.4) Lemma. *Maintain the notation above and assume x_1, \dots, x_m are a permutation of u_1, \dots, u_m . For $\mathbf{a} \in \Lambda[m, r]$, the right ideal $\pi(\mathbf{a})\mathbf{H}$ is spanned by $\mathcal{B}_{\mathbf{a}} = \{\pi(\mathbf{a})L_1^{c_1} \cdots L_r^{c_r} T_w \mid w \in \mathfrak{S}_r, \deg_j \pi(\mathbf{a}) + c_j \leq m-1, \forall j\}$.*

Proof. Let $M_{\mathbf{a}}$ be the R -submodule spanned by $\mathcal{B}_{\mathbf{a}}$. Then $M_{\mathbf{a}} \subseteq \pi(\mathbf{a})\mathbf{H}$. Since $\pi(\mathbf{a}) \in M_{\mathbf{a}}$, it suffices to prove that $M_{\mathbf{a}}$ is a right \mathbf{H} -module, or equivalently $M_{\mathbf{a}}T_0 \subseteq M_{\mathbf{a}}$.

For $\pi(\mathbf{a})L_1^{c_1} \cdots L_r^{c_r} T_w \in \mathcal{B}_{\mathbf{a}}$. We write $w = s_{i,1}y$ with $y \in \mathfrak{S}_{\{2, \dots, r\}}$ (cf. (1.3)). Since $M_{\mathbf{a}}$ is a right \mathcal{H} -module and $T_0 T_y = T_y T_0$, we only need to prove $\pi(\mathbf{a})L_1^{c_1} \cdots L_r^{c_r} T_{i,1} T_0 \in M_{\mathbf{a}}$, which is equivalent to

$$(4.5) \quad \pi(\mathbf{a})L_1^{c_1} \cdots L_r^{c_r} L_i \in M_{\mathbf{a}} \text{ for every } 1 \leq i \leq r.$$

If $a_{m-1} = 0$, then $\pi(\mathbf{a}) = 1$ and $\pi(\mathbf{a})\mathbf{H} = \mathbf{H}$. By (2.2), (4.5) holds in this case. We now assume $a_{m-1} \neq 0$. Apply induction on i . The last relation in (2.1) implies (4.5) for $i = 1$. (If $\deg_1(\pi(\mathbf{a})) + c_1 + 1 = m$, then $\pi(\mathbf{a})$ contains part of the product in (2.1)(5) involving parameters x_i, \dots, x_{m-1} . Write $L_1^{c_1+1} = \prod_k (L_1 - u_{i_k}) +$

$\sum_{j \leq c_1} \alpha_j L_1^j$, where $\{u_{i_k}\} = \{x_1, \dots, x_{i-1}, x_m\}$. Now, (2.1)(5) implies (4.5) in this case.)

We assume now that $i > 1$ and (4.5) holds for all L_j with $j < i$. The case for $\deg_i \pi(\mathbf{a}) + c_i < m - 1$ is trivial. Suppose $\deg_i \pi(\mathbf{a}) + c_i = m - 1$. Let k be the integer with $a_{k-1} < i \leq a_k$. Then, $\deg_i \pi(\mathbf{a}) = m - k$ and $\pi_{a_j}(x_j)T_{i,1} = T_{i,1}\pi_{a_j}(x_j)$ for all $j \geq k$. On the other hand, by (2.5)(d) and induction on i , we have for some c

$$(4.6) \quad L_i^l - q^c T_{i,1} L_1^l T_{1,i} \in U_{i,l}$$

where $U_{i,l}$ is the free R -submodule spanned by $\{L_1^{d_1} \dots L_i^{d_i} T_w \mid 0 \leq d_j < l, \forall j \leq i, w \in \mathfrak{S}_i\}$. In particular, $L_i^{c_i+1} = q^c T_{i,1} L_1^{c_i+1} T_{1,i} + h$ for some c and $h \in U_{i,c_i+1}$. So we have

$$\begin{aligned} & \pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} L_i^{c_i} L_{i+1}^{c_{i+1}} \dots L_r^{c_r} L_i \\ &= \pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} L_i^{c_i+1} L_{i+1}^{c_{i+1}} \dots L_r^{c_r} \\ &= \pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} (q^c T_{i,1} L_1^{c_i+1} T_{1,i} + h) L_{i+1}^{c_{i+1}} \dots L_r^{c_r} \\ &= q^c \pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} T_{i,1} L_1^{c_i+1} T_{1,i} L_{i+1}^{c_{i+1}} \dots L_r^{c_r} \\ & \quad + \pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} h L_{i+1}^{c_{i+1}} \dots L_r^{c_r} \\ &= X_1 + X_2 \end{aligned}$$

We now prove that each of the last two terms X_i above is in $M_{\mathbf{a}}$. For $h = L_1^{d_1} \dots L_i^{d_i} T_w \in U_{i,c_i+1}$, we have $T_w L_j = L_j T_w$ for all $j \geq i+1$ and $d_i < c_i + 1$, and so, $\deg_i \pi(\mathbf{a}) + d_i < \deg_i \pi(\mathbf{a}) + c_i + 1 = m$. On the other hand, we have, by definition, $\deg_j \pi(\mathbf{a}) + c_j \leq m - 1$ for all $1 \leq j \leq m$. Therefore, by inductive hypothesis,

$$\pi(\mathbf{a}) L_1^{c_1} \dots L_{i-1}^{c_{i-1}} (L_1^{d_1} \dots L_i^{d_i} T_w) L_{i+1}^{c_{i+1}} \dots L_r^{c_r} = \pi(\mathbf{a}) \prod_{j < i} L_j^{c_j+d_j} L_i^{d_i} \prod_{j > i} L_j^{c_j} T_w \in M_{\mathbf{a}}.$$

This proves $X_2 \in M_{\mathbf{a}}$.

Since $i \leq a_k$, we have

$$\begin{aligned} X_1 &= \prod_{j=1}^{i-1} L_j^{c_j} \prod_{j=1}^{k-1} \pi_{a_j}(x_j) \left(T_{i,1} \prod_{j=k}^{m-1} \pi_{a_j}(x_j) L_1^{c_i+1} T_{1,i} \right) \prod_{j=i+1}^r L_j^{c_j} \\ &= X_{11}(X_{12})X_{13}. \end{aligned}$$

where X_{11} (resp. X_{13}) denotes the first two products (resp. last product) and X_{12} denotes the element in the parenthesis. As in the argument for $i = 1$, $L_1^{c_i+1} \prod_{j=k}^{m-1} \pi_{a_j}(x_j)$ is an R -linear combinations of $L_1^l \prod_{j=k}^{m-1} \pi_{a_j}(x_j)$ with $l < c_i + 1$, while, by (4.6), $L_i^l - T_{i,1} L_1^l T_{1,i} = h \in U_{i,l}$. Thus, $T_{i,1} L_1^l \prod_{j=k}^{m-1} \pi_{a_j}(x_j) T_{1,i} = \prod_{j=k}^{m-1} \pi_{a_j}(x_j) L_i^l + \prod_{j=k}^{m-1} \pi_{a_j}(x_j) h$. Hence X_1 can be expressed as a linear combination of $X_{11}(\prod_{j=k}^{m-1} \pi_{a_j}(x_j) L_i^l) X_{13}$ and $X_{11}(\prod_{j=k}^{m-1} \pi_{a_j}(x_j) h) X_{13}$, where $l \leq c_i$ and $h \in U_{i,l}$. Since $l \leq c_i$, we have clearly $X_{11}(\prod_{j=k}^{m-1} \pi_{a_j}(x_j) L_i^l) X_{13} \in M_{\mathbf{a}}$. By

inductive hypothesis, we have $X_{11}(\pi_{a_j}(x_j)h)X_{13} \in M_{\mathbf{a}}$, too. Therefore, $X_1 \in M_{\mathbf{a}}$, proving (4.5). \square

For $\mathbf{a} = [a_i] \in \Lambda[m, r]$, let \mathbf{a}_i be defined as in (1.9). Recall $h_{\mathbf{a}} = T_{w_{\mathbf{a}}}$.

(4.7) Proposition. *For $\mathbf{a} \in \Lambda[m, r]$, write $\mathbf{a}_{\leftarrow} = [b_0, \dots, b_m]$ as in (1.10). Let $V_i = \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} \mathbf{H}$.*

Then

(a) $V_1 = v_{\mathbf{a}_{\leftarrow}} \mathbf{H}$ and $V_m = v_{\mathbf{a}_m} \mathbf{H}$.

(b) *The set $\mathcal{B}_i = \{\pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_{r-b_{i-1}}^c T_w \mid c \leq m-i, w \in \mathfrak{S}_r\}$ is a basis of V_i . In particular, the rank of V_i is $(m-i+1)r!$.*

(c) V_{i+1} is a pure R -submodule of V_i .

Proof. We first note that $b_i = a_i, 1 \leq i \leq k-1, b_i = r-1, k \leq i \leq m$, where k is the minimal index with $a_k = r$. Note also from (1.9) and (1.2) that

$$\begin{aligned} \mathbf{a}'_1 &= [0, \dots, 0, r - a_{k-1} - 1, \dots, r - a_1 - 1, r] \\ (\mathbf{a}_{\leftarrow})' &= [0, \dots, 0, r - a_{k-1} - 1, \dots, r - a_1 - 1, r - 1]. \end{aligned}$$

Thus, $\tilde{\pi}_{\mathbf{a}'_1} = \tilde{\pi}_{(\mathbf{a}_{\leftarrow})'}$ by definition (2.6), and $V_1 = v_{\mathbf{a}_{\leftarrow}} \mathbf{H}$. Because $h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{m-1}} = h_{\mathbf{a}_m}$ (see (1.10)) and $\pi_{\mathbf{a}_m} = \pi_{\mathbf{a}_{\leftarrow}}$, we have $V_m = v_{\mathbf{a}_m} \mathbf{H}$, proving (a).

For (b), we first prove that \mathcal{B}_i spans V_i . Let V'_i be the submodule spanned by \mathcal{B}_i . Applying (4.4) to $\tilde{\pi}_{\mathbf{a}'_i} \mathbf{H}$, it suffices to prove

$$(4.8) \quad \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_1^{c_1} \cdots L_{r-b_{i-1}}^{c_{r-b_{i-1}}} \cdots L_r^{c_r} \in V'_i,$$

where $c_j \leq m-1 - \deg_j \tilde{\pi}_{\mathbf{a}'_i}, \forall j$. Since $\deg_{r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} = i-1$, we have, in particular, $c_{r-b_{i-1}} \leq m-i$.

We first look at the elements $\pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_j$ with $j \neq r-b_{i-1}$. Suppose $j < r-b_{i-1}$ and write $T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} = T_{r, r-b_{i-1}} \tilde{\pi}_{(\mathbf{a}_{\leftarrow})'} (L_{r-b_1} - u_1) \cdots (L_{r-b_{i-1}} - u_{i-1}) = \tilde{\pi}_{(\mathbf{a}_{\leftarrow})'} h_i$, where $h_i = T_{r, r-b_1} (L_{r-b_1} - u_1) \cdots T_{r-b_{i-2}, r-b_{i-1}} (L_{r-b_{i-1}} - u_{i-1})$. Then $L_j h_i = h_i L_j$ and $h_i T_l = T_l h_i$ for all $s_l \in \mathfrak{S}_j$. Thus, by (3.1)(b),

$$(4.9) \quad \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_j = v_{\mathbf{a}_{\leftarrow}} L_j h_i \in \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} \mathcal{H}(\mathfrak{S}_j).$$

Suppose now $j \geq r-b_{i-1}+1$. Choose l such that $r-b_l+1 \leq j \leq r-b_{l-1}$. Then $l \leq i-1$ and l is the minimal index having property $b_l = b_{i-1}$. Using definition (2.6), we have $\tilde{\pi}_{\mathbf{a}'_i} (L_{r-b_l+1} - u_l) = \tilde{\pi}_{\mathbf{c}'}$, where $\mathbf{c}' = [0, b_1, \dots, b_{l-1}, b_l-1, b_{l+1}, \dots, b_m] + \mathbf{1}_i \in \Lambda[m, r]$ (see (1.9)). Since $\pi_{\mathbf{a}_{\leftarrow}} = \pi_{\mathbf{a}_m}$ and $\mathbf{a}_m \not\leq \mathbf{c}'$ (as $l < i$), we have, by (2.8) that $\pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} (L_{r-b_l+1} - u_l) = 0$. Therefore, for some c ,

$$(4.10) \quad \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_j = u_l q^c \pi_{\mathbf{a}_{\leftarrow}} h_{\mathbf{a}_{\leftarrow}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} T_{j, r-b_l+1} T_{r-b_l+1, j}.$$

Now, noting (2.5)(a), we see that (4.8) follows from (4.9) and (4.10), since the elements in $\mathcal{H}(\mathfrak{S}_j)$ occurring in (4.8) and the element $T_{j, r-b_l+1} T_{r-b_l+1, j}$ occurring in (4.9) commute with L_{j+1} and $L_{r-b_{i-1}}$.

To see the linear independence, we apply induction on i . For $i = 1$, suppose

$$\sum_{\substack{0 \leq c \leq m-1 \\ w \in \mathfrak{S}_r}} f_{c,w} v_{\mathbf{a}_{\leftarrow}} L_r^c T_w = 0.$$

Since $pr_h(v_{\mathbf{a}_{-1}}) = q^d \prod_{j=1}^{r-1} L_j^{m-1} h_{(\mathbf{a}_{-1})'}^{-1}$, for some $d \in \mathbb{Z}$ depending on \mathbf{a}_{-1} (see (3.5)), we have

$$pr_{\mathbf{c}}\left(\sum_{\substack{0 \leq c \leq m-1 \\ w \in \mathfrak{S}_r}} f_{c,w} v_{\mathbf{a}_{-1}} L_r^c T_w\right) = q^d \sum_{w \in \mathfrak{S}_r} f_{c,w} L_1^{m-1} \cdots L_{r-1}^{m-1} L_r^c h_{(\mathbf{a}_{-1})'}^{-1} T_w = 0,$$

where $\mathbf{c} = (m-1, \dots, m-1, c)$. Thus, $f_{c,w} = 0$ for all c and w , since both q and $h_{(\mathbf{a}_{-1})'}^{-1}$ are invertible. Therefore, \mathcal{B}_1 is a basis of V_1 .

We assume now that \mathcal{B}_i is a linearly independent set. We hope to prove that the set \mathcal{B}_{i+1} is linearly independent, too. Using induction on l , we can easily prove

$$T_{a,b} L_b^l = q^{a-b} L_a^l T_{b,a}^{-1} + \text{terms involving } L_a^{c_a} \cdots L_b^{c_b} T_w \text{ for } a \geq b$$

where $w \in \mathfrak{S}_{\{b, b+1, \dots, a\}}$ and all $c_i < l$. Using this and noting $\tilde{\pi}_{\mathbf{a}'_{i+1}} = \tilde{\pi}'_{\mathbf{a}_i}(L_{r-b_i} - u_i)$, we have

$$(4.11) \quad \begin{aligned} & \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_i} \tilde{\pi}_{\mathbf{a}'_{i+1}} L_{r-b_i}^c T_w \\ &= q^{a_i - a_{i-1}} \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_{r-b_{i-1}}^{c+1} T_{r-b_i, r-b_{i-1}}^{-1} T_w + \\ & \quad + \text{terms involving } \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_{r-b_{i-1}}^l T_z, \end{aligned}$$

where $l < c+1$ and $z \in \mathfrak{S}_r$. By induction, \mathcal{B}_i is a basis of V_i . Because $T_{r-b_i, r-b_{i-1}}$ is invertible, the set $\{\pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} L_{r-b_{i-1}}^c T_{r-b_i, r-b_{i-1}}^{-1} T_w \mid c \leq m-i, w \in \mathfrak{S}_r\}$ is linearly independent in V_i . Therefore, the condition $l < c+1$ in (4.11) implies that \mathcal{B}_{i+1} is linear independent. This completes the proof of (4.7)(b).

We now prove the purity in (c). Since

$$\pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_i} \tilde{\pi}_{\mathbf{a}'_{i+1}} = \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} T_{r-b_{i-1}, r-b_i} (L_{r-b_i} - u_i),$$

V_{i+1} is a submodule of V_i . Write $h = \sum f_{c,w} \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_i} \tilde{\pi}_{\mathbf{a}'_{i+1}} L_{r-b_i}^c T_w$ for $h \in V_{i+1}$. To show that V_{i+1} is a pure R -submodule, we need to show that $x \in R$ divides any of $f_{c,w}$ if $h \in xV_i$. We prove it by induction on c_0 , the highest degree of L_{r-b_i} in the expression of h . Since \mathcal{B}_i is a basis of V_i and q is invertible, x divides $f_{c_0, w}$ by (4.11). Thus, $h - \sum f_{w, c_0} \pi_{\mathbf{b}} h_{\mathbf{b}} T_{r, r-b_i} \tilde{\pi}_{\mathbf{a}'_{i+1}} L_{r-b_i}^c T_w \in xV_i$. Now, the result follows from induction. \square

From here onwards, we assume that R is an *integral domain*.

(4.12) Lemma. *Keep the notation above. We have short exact sequence*

$$0 \rightarrow V_{i+1} \rightarrow V_i \rightarrow v_{\mathbf{a}_i} \mathbf{H} \rightarrow 0.$$

Proof. Let

$$h_i = (L_{b_{i+1}} - u_{i+1}) T_{b_{i+1}, b_{i+1}+1} \cdots (L_{b_{k-1}+1} - u_k) T_{b_{k-1}+1, r} \prod_{l=k+1}^m (L_r - u_l),$$

where, as usual, k is the minimal index such that $a_k = r$. Then, by (2.5)(a)-(c), we have $h_i \pi_{\mathbf{a}_{-1}} = \pi_{\mathbf{a}_i} T_{b_{i+1}, r}$. Define $\psi_i : V_i \rightarrow v_{\mathbf{a}_i} \mathbf{H}$ by setting $\psi_i(h) = h_i h$, $h \in V_i$. Note that ψ_i is well-defined by (1.10), and clearly, ψ_i is surjective. Because $\mathbf{a}_i \succ \mathbf{a}_{i+1}$, we have $V_{i+1} \subseteq \ker \psi_i$ by (2.8) and (3.1)(c). By (3.4) and (4.7)(b), we have $\dim V_i = \dim V_{i+1} + \dim v_{\mathbf{a}_i} \mathbf{H}$, forcing $V_{i+1} = \ker \psi_i$ over the quotient field F of R . However, by (4.7)(c), V_{i+1} is a pure R -submodule of V_i . Thus $V_{i+1} = \ker \psi_i$, proving the required short exact sequence. \square

(4.13) Lemma. *Let $\mathbf{a} = [a_i] \in \Lambda[m, r]$ and write $\mathbf{a}_{-1} = [b_i]$. Assume k is the minimal index with $a_k = r$. Define*

$$\tilde{V}_i = \pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} \prod_{j=i+1}^m \pi_{r-b_{i-1}}(u_j) \mathbf{H}, \text{ for } i = k, \dots, m-1.$$

If $f_{m,r}$ is a unit in the integral domain R , then we have $V_i = V_{i+1} \oplus \tilde{V}_i$ and $\tilde{V}_i \cong v_{\mathbf{a}_i} \mathbf{H}$ for all $i = k, \dots, m-1$.

Proof. Let $\psi_i : V_i \rightarrow v_{\mathbf{a}_i} \mathbf{H}$, $i \geq k$, be the \mathbf{H} -module homomorphism defined in the proof of (4.12). Then, $\psi_i(\tilde{V}_i) = v_{\mathbf{a}_i} \prod_{j=i+1}^m \pi_{r-b_{i-1}}(u_j) \mathbf{H}$. By (3.1)(b), we have $v_{\mathbf{a}_i} L_1 = u_i v_{\mathbf{a}_i}$ for all $i \geq k$. Also, for $j < r - a_{k-1}$, $s_j \in \mathfrak{S}_{\mathbf{a}'_k}$. So $v_{\mathbf{a}_k} T_j = T_{j'} v_{\mathbf{a}_k}$ (see (3.1)(a)), and hence, $v_{\mathbf{a}_k} L_l = u_k q^{1-l} v_{\mathbf{a}_k} T_{l,1} T_{1,l}$ for all $l = 1, \dots, r - b_{k-1}$, noting $a_{k-1} = b_{k-1}$. Therefore, we have

$$\psi_i(\tilde{V}_i) = v_{\mathbf{a}_i} \prod_{j=i+1}^m \pi_{r-b_{i-1}}(u_j) \mathbf{H} = v_{\mathbf{a}_i} \prod_{j=i+1}^m \prod_{l=1}^{r-b_{i-1}} (q^{1-l} u_i T_{l,1} T_{1,l} - u_j) \mathbf{H}.$$

Since $\prod_{j=i+1}^m \prod_{l=1}^{r-b_{i-1}} (q^{1-l} u_i T_{l,1} T_{1,l} - u_j)$ is a factor of $z_{\mathbf{r}'_i}$ (see (4.1)), and $f_{m,r} \in R$ is a unit, $\prod_{l=1}^{r-b_{i-1}} (q^{1-l} u_i T_{l,1} T_{1,l} - u_j)$ is invertible by (4.3). Therefore, $\psi_i(\tilde{V}_i) = v_{\mathbf{a}_i} \mathbf{H}$. Thus, by (4.12), the rows of the following commutative diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & V_{i+1} & \rightarrow & \tilde{V}_i + V_{i+1} & \xrightarrow{\psi_i} & v_{\mathbf{a}_i} \mathbf{H} \rightarrow 0 \\ & & \parallel & & \downarrow & & \parallel \\ 0 & \rightarrow & V_{i+1} & \rightarrow & V_i & \xrightarrow{\psi_i} & v_{\mathbf{a}_i} \mathbf{H} \rightarrow 0. \end{array}$$

are exact. So the short-five lemma implies $\tilde{V}_i + V_{i+1} = V_i$. On the other hand, consider the element $x = \tilde{\pi}_{\mathbf{a}'_i} \prod_{j=i+1}^m \pi_{r-b_{i-1}}(u_j)$ and the multiplication $x L_{r-b_{i-1}}$. Clearly, x has a factor $(L_1 - u_1) \cdots (L_1 - u_{i-1})(L_1 - u_{i+1}) \cdots (L_1 - u_m)$. So, if $i > k$, then $r - b_{i-1} = 1$, and hence, $x L_1 = u_i x$, while, for $i = k$, we have $x T_{r-b_{k-1}, 1} = T_{r-b_{k-1}, 1} x$ and consequently, $x L_{r-b_{k-1}} = q^c u_i x T_{r-b_{k-1}, 1} T_{1, r-b_{k-1}}$. Therefore, by (4.7)(b), \tilde{V}_i is spanned by $\{\pi_{\mathbf{a}_{-1}} h_{\mathbf{a}_{-1}} T_{r, r-b_{i-1}} \tilde{\pi}_{\mathbf{a}'_i} \prod_{j=i+1}^m \pi_{r-b_{i-1}}(u_j) T_w \mid w \in \mathfrak{S}_r\}$, which is linearly independent, since its image under the homomorphism ψ_i is linearly independent. Hence, \tilde{V}_i is R -free of rank $r!$, forcing the sum $\tilde{V}_i + V_{i+1}$ is a direct sum, i.e., $V_i = \tilde{V}_i \oplus V_{i+1}$, and consequently, $\tilde{V}_i \cong v_{\mathbf{a}_i} \mathbf{H}$. \square

We now prove the main result of the paper. The case when \mathbf{H} is the Hecke algebra of type B was first obtained in [DJ2].

(4.14) Theorem. *Let $\mathbf{H} = \mathbf{H}_m^r$ be the Ariki-Koike algebra over an integral domain R . Assume $f_{m,r} \in R$ is a unit.*

- (a) *For every $\mathbf{a} \in \Lambda[m, r_1]$ with $r_1 \leq r$, the right ideal $v_{\mathbf{a}}\mathbf{H}$ is projective.*
- (b) *For any given $r_1 \leq r$, each projective indecomposable \mathbf{H} -module is isomorphic to a direct summand of $v_{\mathbf{a}}\mathbf{H}$ for some $\mathbf{a} \in \Lambda[m, r_1]$.*
- (c) *The categories of \mathbf{H} -modules and $\bigoplus_{\lambda \in \Lambda(m,r)} \mathcal{H}(\mathfrak{S}_\lambda)$ -modules are Morita equivalent.*

Proof. For $r_1 \leq r$, \mathbf{H} is free over $\mathbf{H}_m^{r_1}$. So the functor $-\otimes_{\mathbf{H}_m^{r_1}} \mathbf{H}$ is exact. Therefore, it suffices to prove (a) for $r_1 = r$.

We apply induction on r . For $r = 1$, we have $\mathbf{a} = \mathbf{r}_i$ for some i . By the invertibility of $f_{m,r}$, (4.3) and (3.8), we see that $v_{\mathbf{a}}\mathbf{H}$ is projective. Assume now that $r > 1$ and the result holds for $r - 1$. We now prove the projectivity of $v_{\mathbf{a}}\mathbf{H}$ by downward induction on the partial ordering \succ . If $\mathbf{a} = \mathbf{r}_1 = [0, r, \dots, r]$, the maximal element of $\Lambda[m, r]$, then $v_{\mathbf{a}}\mathbf{H}$ is projective by (4.3) and (3.8). Suppose now that $\mathbf{r}_1 \succ \mathbf{a}$ and that $v_{\mathbf{b}}\mathbf{H}$ is projective for all $\mathbf{b} \succ \mathbf{a}$. Let \mathbf{a}_{\leftarrow} and \mathbf{a}_i , $1 \leq i \leq m$ be defined as in (1.2) and (1.9), and k the minimal index with $a_k = r$. Then $k \geq 2$ and $\mathbf{a}_k = \mathbf{a}$. By induction, $v_{\mathbf{a}_j}\mathbf{H}$, $1 \leq j \leq k - 1$, are projective, and $v_{\mathbf{a}_{\leftarrow}}\mathbf{H}$ is projective. Using the short exact sequence in (4.12), we have $v_{\mathbf{a}_{\leftarrow}}\mathbf{H} \cong \bigoplus_{j=1}^{k-1} v_{\mathbf{a}_j}\mathbf{H} \oplus V_k$ and hence, V_k is projective. By (4.13), we have $V_k \cong V_{k+1} \oplus v_{\mathbf{a}_k}\mathbf{H}$, and consequently, $v_{\mathbf{a}}\mathbf{H} = v_{\mathbf{a}_k}\mathbf{H}$ is projective, proving (a). Note from (4.7) and (4.12-3) that we have for any $r > 0$ and $\mathbf{a} \in \Lambda[m, r]$

$$(4.15) \quad v_{\mathbf{a}_{\leftarrow}}\mathbf{H} \cong \bigoplus_{i=1}^m v_{\mathbf{a}_i}\mathbf{H}.$$

To see (b), we apply the tensor functor $-\otimes_{\mathbf{H}_m^{r_1}} \mathbf{H}$ to (4.15) and obtain, for $1 \leq r_1 \leq r$ and $\mathbf{a} \in \Lambda[m, r_1]$

$$v_{\mathbf{a}_{\leftarrow}}\mathbf{H} \cong \bigoplus_{j=1}^m v_{\mathbf{a}_j}\mathbf{H},$$

from which (b) follows.

Putting $r_1 = r$ in both (a) and (b), we see that the \mathbf{H} -module $\bigoplus_{\mathbf{a} \in \Lambda[m, r]} v_{\mathbf{a}}\mathbf{H}$ is a projective generator. By standard results (see, e.g., [CR, (3.54)]) the categories of \mathbf{H} -modules and $\varepsilon\mathbf{H}\varepsilon$ -modules are Morita equivalent (cf. (3.11)). \square

5. Some applications. We first recall the notion of multi-compositions and multi-partitions of r . By definition, $\boldsymbol{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is called an m -composition (resp. m -partition) of r if $\lambda^{(i)}$ is a composition (resp. partition) for every i with $1 \leq i \leq m$ and $|\boldsymbol{\lambda}| = \sum_{i=1}^m |\lambda^{(i)}| = r$. Recall that $\Lambda(n, r)$ (resp. $\Lambda(n, r)^+$) is the set of all compositions (resp. partitions) of r with n parts, and let, for $m > 0$ and $\mu = (\mu_1, \dots, \mu_m) \in \Lambda(m, r)$,

$$\begin{aligned} \Lambda(n, \mu) &= \Lambda(n_r, \mu_1) \times \dots \times \Lambda(n_r, \mu_{m-1}) \times \Lambda(n, \mu_m) \\ \Lambda_m(n, r) &= \bigcup_{\mu \in \Lambda(m, r)} \Lambda(n, \mu) \end{aligned}$$

where n_r is the maximum of n and r . Define $\Lambda_m(n, r)^+$ similarly. The set $\Lambda_m(n, r)$ can be identified with $\Lambda(N, r)$ by concatenation, where $N = (m - 1)n_r + n$. To

distinguish them, for $\lambda \in \Lambda_m(n, r)$, let $\bar{\lambda}$ denote the corresponding element in $\Lambda(N, r)$.

For $\lambda \in \Lambda_m(n, \mu)$, let $\mathbf{a} = [a_0, \dots, a_m]$ be the cumulative norm sequence of λ , denoted $\text{cns}(\lambda)$, where $a_0 = 0$ and $a_i = \mu_1 + \dots + \mu_i$ for $1 \leq i \leq m$. Let e be the minimal integer l such that

$$1 + q + q^2 + \dots + q^{l-1} = 0.$$

If such an integer l does not exist, then set $e = \infty$. A partition λ is called e -regular if λ has no non-zero part occurring e or more times. An m -partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ is e -regular if each $\lambda^{(i)}$, $1 \leq i \leq m$, is e -regular.

Let \mathcal{H}_F be the Hecke algebra of type A_{r-1} over a field F , in which q is a primitive e -th root of 1. Let S^λ be the Specht module with respect to λ . Then, by [DJ1], S^λ has simple head D^λ , if λ is e -regular. (Note, for $e > r$, $D^\lambda = S^\lambda$.) For a multi-partition $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)})$ of r with $\text{cns}(\lambda) = \mathbf{a}$, let $S^{\lambda^{(1)}} \dots S^{\lambda^{(m)}} (\cong \otimes_{i=1}^m S^{\lambda^{(i)}})$ be the corresponding Specht module for $\mathcal{H}_F(\mathfrak{S}_{\mathbf{a}})$, and let S^λ be the right ideal $e_{\mathbf{a}} S^{\lambda^{(1)}} \dots S^{\lambda^{(m)}} \mathbf{H}_F$ of \mathbf{H}_F . By (4.14)(c), we know that the \mathbf{H} -module S^λ and $\varepsilon \mathbf{H} \varepsilon$ -module $S^\lambda \varepsilon$ have isomorphic submodule lattices. Since $S^\lambda \varepsilon \cong S^{\lambda^{(1)}} \dots S^{\lambda^{(m)}}$ and the latter has simple head if λ is e -regular, so is S^λ . Let D^λ denote the simple head of S^λ . The following is an immediate consequence of (4.14)(c) (see [DJ2, (5.3)] for the case $m = 2$).

(5.1) Theorem. *Let F be a field and $f_{m,r} \neq 0$ in F . Then the set*

$$\{D^\lambda \mid \lambda \in \Lambda_m(r, r)^+ \text{ } e\text{-regular}\}$$

is a complete set of simple \mathbf{H}_F -modules.

The following result has been proved in [Ari]. Recall the Poincaré polynomial d_W introduced in (4.2)

(5.2) Theorem. *Let \mathbf{H}_F be the Ariki-Koike algebra over a field F . Then \mathbf{H}_F is semisimple if and only if $d_W \neq 0$.*

Proof. Note that $d_W \neq 0$ is equivalent to $f_{m,r} \neq 0$ and $e > r$. By (4.3), we have $f_{m,r} \neq 0$ if and only if $f_{m,r,i} \neq 0$ for all i . If $f_{m,r} \neq 0$, then the categories of \mathbf{H} -modules and $\varepsilon \mathbf{H} \varepsilon$ -modules are Morita equivalent. By (3.10) and [DJ1, 4.3], $\varepsilon \mathbf{H} \varepsilon$ is semisimple if and only if $e > r$, proving the “if” part. Conversely, it suffices to look at the case $f_{m,r} = 0$. Then $f_{m,i,r} = 0$ for some i , and $z_{r'_i}$ is not invertible. Therefore, $v_{r'_i} \mathbf{H}$ is not an idempotent ideal by (3.9). Thus, \mathbf{H} is not semisimple. \square

We finally look at a Morita theorem between q -Schur ^{m} algebras and q -Schur algebras (i.e., the q -Schur¹ algebras). For any $\lambda \in \Lambda_m(n, \mathbf{r})$ with $\text{cns}(\lambda) = \mathbf{a}$, let $x_\lambda = \pi_{\mathbf{a}} x_{\bar{\lambda}}$.

(5.3) Lemma. *Let R be an integral domain and assume that $f_{m,r} \in R$ is a unit. For any $\lambda \in \Lambda_m(n, r)$ with $\text{cns}(\lambda) = \mathbf{a}$, we have $e_{\mathbf{a}} x_\lambda \mathbf{H} = e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H}$.*

Proof. Since $\pi_{\mathbf{a}} \mathbf{H} \subseteq \mathbf{H}$, we have $e_{\mathbf{a}} x_\lambda \mathbf{H} \subseteq e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H}$. By (1.8), (3.1)(a) and (3.6)-(3.7), we have $e_{\mathbf{a}} x_{\bar{\lambda}} = v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} x_{\bar{\lambda}} = v_{\mathbf{a}} h_{\mathbf{a}'} x_{\bar{\lambda}} z_{\mathbf{a}}^{-1} = v_{\mathbf{a}} h h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} = x_{\bar{\lambda}} v_{\mathbf{a}} h_{\mathbf{a}'} z_{\mathbf{a}}^{-1} =$

$x_{\bar{\lambda}}e_{\mathbf{a}}$, where $h \in \mathcal{H}(\mathfrak{S}_{\mathbf{a}'})$ given by $hh_{\mathbf{a}'} = h_{\mathbf{a}'}x_{\bar{\lambda}}$. Thus, $e_{\mathbf{a}}x_{\lambda}\mathbf{H} = x_{\bar{\lambda}}e_{\mathbf{a}}\pi_{\mathbf{a}}\mathbf{H} = x_{\bar{\lambda}}v_{\mathbf{a}}h_{\mathbf{a}'}\pi_{\mathbf{a}}\mathbf{H} \supseteq x_{\bar{\lambda}}v_{\mathbf{a}}h_{\mathbf{a}'}v_{\mathbf{a}}\mathbf{H} = x_{\bar{\lambda}}v_{\mathbf{a}}z_{\mathbf{a}'}\mathbf{H} = x_{\bar{\lambda}}e_{\mathbf{a}}\mathbf{H}$, proving (5.3). \square

Some special case for $m = 2$ of the second part of the following result has been discussed by Gruber and Hiss [GH] in the context of representations finite groups of Lie type.

(5.4) Corollary. *Let \mathbf{H} be the Ariki-Koike algebra over an integral domain R , in which $f_{m,r}$ is a unit. Let $\mathbf{a} = \text{cns}(\lambda)$ for $\lambda \in \Lambda_m(n, r)$. Then*

$$\begin{aligned} \text{End}_{\mathbf{H}} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\lambda} \mathbf{H} \right) &= \text{End}_{\mathbf{H}} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H} \right) \\ &\cong \bigoplus_{\mu \in \Lambda(m, r)} \text{End}_{\mathcal{H}(\mathfrak{S}_{\mu})} \left(\bigoplus_{\lambda \in \Lambda(n, \mu)} x_{\bar{\lambda}} \mathcal{H}(\mathfrak{S}_{\mu}) \right). \end{aligned}$$

Proof. The first equality follows from (5.3). Using standard results (see [AF, (21.2)] or [DPS, (0.1)]), we know that

$$\text{End}_{\mathbf{H}} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H} \right) \cong \text{End}_{\varepsilon \mathbf{H} \varepsilon} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H} \varepsilon \right).$$

For $\lambda = (\lambda^{(1)}, \dots, \lambda^{(m)}) \in \Lambda_m(n, r)$, write $\text{cns}(\lambda) = \mathbf{a} = [a_i]$. Then we have $x_{\bar{\lambda}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) \cong x_{\lambda^{(1)}} \mathcal{H}(\mathfrak{S}_{a_1}) \otimes_R \dots \otimes_R x_{\lambda^{(m)}} \mathcal{H}(\mathfrak{S}_{a_m - a_{m-1}})$, and by the proof of (5.3), $e_{\mathbf{a}} x_{\bar{\lambda}} = x_{\bar{\lambda}} e_{\mathbf{a}}$. Therefore, $e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H} \varepsilon = x_{\bar{\lambda}} e_{\mathbf{a}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}}) \cong x_{\bar{\lambda}} \mathcal{H}(\mathfrak{S}_{\mathbf{a}})$. By (3.10), we have

$$\begin{aligned} \text{End}_{\varepsilon \mathbf{H} \varepsilon} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\bar{\lambda}} \mathbf{H} \varepsilon \right) &\cong \bigoplus_{\mathbf{a} \in \Lambda[m, r]} \text{End}_{e_{\mathbf{a}} \mathbf{H} e_{\mathbf{a}}} \left(\bigoplus_{\lambda \in \Lambda(n, \Theta(\mathbf{a}))} x_{\bar{\lambda}} e_{\mathbf{a}} \mathbf{H} e_{\mathbf{a}} \right) \\ &\cong \bigoplus_{\mu \in \Lambda(m, r)} \text{End}_{\mathcal{H}(\mathfrak{S}_{\mu})} \left(\bigoplus_{\lambda \in \Lambda(n, \mu)} x_{\bar{\lambda}} \mathcal{H}(\mathfrak{S}_{\mu}) \right). \end{aligned}$$

Here Θ is defined in (1.1). \square

The endomorphism algebra $\mathbf{S}_R^m(n, r) = \text{End}_{\mathbf{H}} \left(\bigoplus_{\lambda \in \Lambda_m(n, r)} x_{\lambda} \mathbf{H} \right)$ is called a q -Schur ^{m} algebra in [DR2], which is a cyclotomic q -Schur algebra in the sense of [DJM]. It is proved in [DR2] that a Borel type subalgebra of a q -Schur ^{m} algebra is isomorphic to a Borel subalgebra of a q -Schur algebra. Now, with the invertibility of the polynomial $f_{m,r}$, we establish below a Morita equivalence between the module categories of a q -Schur ^{m} algebra and a direct sum of tensor products of certain q -Schur algebras.

(5.5) Theorem. *Let $\mathbf{H} = \mathbf{H}_{\mathcal{O}}$ be the Ariki-Koike algebra over a discrete valuation ring \mathcal{O} . Then the q -Schur ^{m} algebra $\mathbf{S}_{\mathcal{O}}^m(n, r)$ is Morita equivalent to the algebra*

$$\bigoplus_{\mu \in \Lambda(m, r)} \mathbf{S}_{\mathcal{O}}^1(n_r, \mu_1) \otimes \dots \otimes \mathbf{S}_{\mathcal{O}}^1(n_r, \mu_{m-1}) \otimes \mathbf{S}_{\mathcal{O}}^1(n, \mu_m).$$

Proof. We first note the isomorphism

$$(5.6) \quad \bigoplus_{\mu} \text{End}_{\mathcal{H}_{\mathcal{O}}(\mathfrak{S}_{\mu})} \left(\bigoplus_{\lambda \in \Lambda(n, \mu)} x_{\bar{\lambda}} \mathcal{H}_{\mathcal{O}}(\mathfrak{S}_{\mu}) \right) \cong \bigoplus_{\mu} \bigotimes_{i=1}^{m-1} \mathbf{S}_{\mathcal{O}}^1(n_r, \mu_i) \otimes \mathbf{S}_{\mathcal{O}}^1(n, \mu_m),$$

where $\mu \in \Lambda(m, r)$. The q -Schur algebra of bidegree (n, r) is a (integral) quasi-hereditary algebra, whose simple modules are parametrized by $\Lambda(n, r)^+$ (see [DS, §2]). Therefore, by [Wi, (1.3)] (or a cellular basis argument [GL], [DR1]), the algebra

$$\bigoplus_{\mu \in \Lambda(m, r)} \bigotimes_{i=1}^{m-1} \mathbf{S}_{\mathcal{O}}^1(n_r, \mu_i) \otimes \mathbf{S}_{\mathcal{O}}^1(n, \mu_m)$$

is a quasi-hereditary algebra, whose simple modules are indexed by $\Lambda_m(n, r)^+$. Thus, PIMs are indexed by $\Lambda_m(n, r)^+$, too. Using Fitting's Lemma, the non-isomorphic indecomposable direct summands (The existence of these modules follows from Heller's result, [CR, (30.18iii)].) of $\bigoplus_{\lambda \in \Lambda_m(n, r)} x_{\bar{\lambda}} e_{\mathbf{a}} \mathbf{H}_{\mathcal{O}} e_{\mathbf{a}}$ are indexed by $\Lambda_m(n, r)^+$. Therefore, the non-isomorphic indecomposable direct summands of $\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\lambda} \mathbf{H}_{\mathcal{O}}$ are indexed by $\Lambda_m(n, r)^+$ by (5.4). Since $e_{\mathbf{a}} x_{\bar{\lambda}} = x_{\bar{\lambda}} e_{\mathbf{a}}$ and $e_{\mathbf{a}}^2 = e_{\mathbf{a}}$, we have $e_{\mathbf{a}} x_{\lambda} \mathbf{H}_{\mathcal{O}} \oplus (1 - e_{\mathbf{a}}) x_{\lambda} \mathbf{H}_{\mathcal{O}} = x_{\lambda} \mathbf{H}_{\mathcal{O}}$. It follows that every direct summand of $e_{\mathbf{a}} x_{\lambda} \mathbf{H}_{\mathcal{O}}$ is a direct summand of $x_{\lambda} \mathbf{H}_{\mathcal{O}}$. Now, by the quasi-heredity of the q -Schur^m algebra (see [DR2, (5.10)]), the non-isomorphic indecomposable direct summands of $\bigoplus_{\lambda \in \Lambda_m(n, r)} x_{\lambda} \mathbf{H}_{\mathcal{O}}$ are indexed by $\Lambda_m(n, r)^+$. Therefore, both $\bigoplus_{\lambda \in \Lambda_m(n, r)} x_{\lambda} \mathbf{H}_{\mathcal{O}}$ and $\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\lambda} \mathbf{H}_{\mathcal{O}}$ have the same non-isomorphic indecomposable direct summands. Consequently, the q -Schur^m algebra $\mathbf{S}_{\mathcal{O}}^m(n, r)$ is Morita equivalent to $\text{End}_{\mathbf{H}_{\mathcal{O}}}(\bigoplus_{\lambda \in \Lambda_m(n, r)} e_{\mathbf{a}} x_{\lambda} \mathbf{H}_{\mathcal{O}})$. Now, the required Morita equivalence follows from (5.4) and (5.6). \square

REFERENCES

- [AF] F.W. Anderson and K.R. Fuller, *Rings and Categories of Modules*, GTM 13, Springer-Verlag, New York, 1973.
- [Ari] S. Ariki, *On the Semi-simplicity of the Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$* , J. Algebra **169** (1994), 216-225.
- [AK] S. Ariki and K. Koike, *A Hecke algebra of $(\mathbb{Z}/r\mathbb{Z}) \wr \mathfrak{S}_n$ and the construction of its irreducible representations*, Adv. Math. **106** (1994), 216-243.
- [CR] C.W. Curtis and I. Reiner, *Methods of Representation Theory with Application to Finite Groups and Orders, Vol I*, Wiley, New York, 1987.
- [DJ1] R. Dipper and G. D. James, *Representations of Hecke algebras of general linear groups*, Proc. London Math. Soc. **52** (1986), 20-52.
- [DJ2] R. Dipper and G. D. James, *Representations of Hecke algebras of type B_n* , J. Algebra **146** (1992), 454-481.
- [DJM] R. Dipper, G. D. James and A. Mathas, *Cyclotomic q -Schur algebras*, Math. Z. **229** (1998), 385-416.
- [DPS] J. Du, B. Parshall and L. Scott, *Quantum Weyl reciprocity and tilting modules*, Comm. Math. Physics **195** (1998), 321-352.
- [DR1] J. Du and H. Rui, *Based algebras and standard bases for quasi-hereditary algebras*, Trans. Amer. Math. Soc. **350** (1998), 3207-3235.
- [DR2] J. Du and H. Rui, *Borel type subalgebras for the q -Schur^m algebras*, J. Algebra (to appear).
- [DS] J. Du and L. Scott, *Lusztig conjectures, old and new, I*, J. reine angew. Math. **455** (1994), 141-182.
- [GH] J. Gruber and G. Hiss, *Decomposition numbers of finite classical groups for linear primes*, J. reine angew. Math. **485** (1997), 55-91.
- [GL] J. Graham and G. Lehrer, *Cellular algebras*, Invent. Math. **126** (1996), 1-34.
- [Hum] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics, Vol 29. Cambridge University Press, Cambridge (1990).
- [MM] A. Mathas and G. Malle, *Symmetric cyclotomic Hecke algebras*, J. Algebra **205** (1998), 275-293.
- [Wi] A. Wiedemann, *On stratifications of derived module categories*, Canad. Math. Bull. **34** (1991), 275-280.